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The Definitions of Fundamental Geometric Entities Contained in Book I of Euclid's Elements

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Abstract

The thesis is sustained that the definitions of fundamental geometric entities which open Euclid's *Elements* actually are excerpts from the *Definitions* by Heron of Alexandria, interpolated in late antiquity into Euclid's treatise. As a consequence, one of the main bases of the traditional Platonist interpretation of Euclid is refuted. Arguments about the constructivist nature of Euclid's mathematical philosophy are given.

Introduction

In Sect. 1 the issue of the authenticity of the first seven definitions of Book I of the *Elements* is introduced. The scant evidence furnished by papyri and a passage of SEXTUS EMPIRICUS are examined in Sects. 2 and 3 respectively. The analysis of the latter suggests the possibility that the seven definitions are excerpts from HERON's *Definitions*, interpolated into EUCLID's text in late antiquity. The close connection between the two series of definitions is shown in Sect. 4 and the plausibility of the above conjecture is examined in Sect. 5 in the light of available information on the textual tradition of EUCLID and HERON. In Sect. 6 some other definitions included in Book I of the *Elements* are shown to have been probably extracted from HERON's work. In Sect. 7 the methodological reasons for supposing that the first seven definitions have the same origin are exposed. In Sect. 8 some other passages of SEXTUS EMPIRICUS on the subject of geometrical definitions are examined. Section 9 contains an analysis of HERON's text, which furnishes some other arguments supporting our thesis. In Sect. 10 it is shown how the derivation from HERON allows us to explain in a natural way the definition of a straight line included in the *Elements*, which is otherwise hard to understand. After some remarks on other testimonies, contained in Sect. 11, conclusions are drawn in Sect. 12.

The present work is based on a paper written some years ago (L. RUSSO 1992) and differs from it mainly in using wider textual evidence and some new arguments. Since, however, the previous work appeared in Italian, in a philological journal little known to historians of science, instead of writing a set of addenda, I have preferred to write a new independent paper.

1. The first seven definitions of Book I of Euclid's *Elements*

In most of *definitions* (ὅροι) included in EUCLID's *Elements* new mathematical objects are defined in terms of previously introduced mathematical entities. We can also find, however, in the *Elements* a few attempts at "defining" elementary mathematical entities by means of ordinary language. In particular we shall be concerned here with the first seven definitions of Book I, i.e. the definitions of a point, a line, a straight line, a surface and a plane (five terms are defined in seven definitions because points and lines are defined twice). We transcribe them in the text established by HEIBERG (EUCLID, vols. I - V) and in an English translation[1]:

1. σημεῖόν ἐστιν, οὗ μέρος οὐθέν, *a point is that which has no part.*
2. γραμμὴ δὲ μῆκος ἀπλατές, *a line is breadthless length.*
3. γραμμῆς δὲ πέρατα σημεῖα, *the extremities of a line are points.*
4. εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται, *a straight line is* [a line] *which lies uniformly in respect to* [all] *its points*.
5. ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει, *a surface is that which has length and breadth only.*
6. ἐπιφανείας δὲ πέρατα γραμμάι, *the extremities of a surface are lines.*
7. ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται, *a plane surface is* [a surface] *which lies uniformly in respect to* [all] *its straight lines*.

Definitions like these are today considered useless and their inclusion in the *Elements* is usually seen as a serious flaw in the *Elements*. A few authors, on the other hand, have raised doubts of a generic nature on their authenticity[2].

The presence of the above definitions in our manuscripts of the *Elements* is indeed far from warranting their authenticity, in view of the scant reliability of the textual tradition, whose main features we shall briefly review here.

All known manuscripts of the *Elements*, except one, go back to a recension usually attributed to THEON OF ALEXANDRIA (IV century A.D.). We know that THEON had inserted into the *Elements* interpolations of his own hand, since in his *Commentary to the Almagest* he refers to a theorem he had proven and inserted in his edition of the *Elements*, namely a statement added to proposition 33 of Book VI (THEON OF ALEXANDRIA, 50). On the other hand we know what could be meant at his time by a new edition of a scientific treatise: for instance in the case of EUCLID's *Optics* we can note the differences between two notably different versions of the work: the one attributed by Heiberg to Theon and

[1] I have used the translation by T. L. HEATH. Only in the case of definitions 4 and 7 have I given a different translation. The meaning of def. 4 and HEATH's translation of it shall be discussed in Sect. 10. Def. 7 does not require an independent discussion, because it is obviously built following the same pattern.

[2] We should remember, in particular, that LORIA, after having criticized the definitions of Book I of the *Elements*, and in particular the definitions we are here concerned with, remarks: *Tutto ciò induce a supporre che il testo da noi posseduto delle definizioni abbia più del resto subita l'influenza modificatrice dei ricopiatori, in genere più audaci nel ritoccare i preliminari che nel modificare i ragionamenti e costruzioni e consiglia di essere cauti nell'imputare ad Euclide le imperfezioni che vi si trovano e non soverchiamente timidi nel toglierle* (LORIA, 202).

an edition identified by him as the "genuine" Euclid's text[3] (both versions are published in EUCLID, vol. VII).

The only manuscript of the *Elements* containing a different recension from the "Theonin" one was found by PEYRARD in the Vatican Library. HEIBERG (1888) maintained that this edition (considered by him an earlier version[4]), should be dated back to not earlier than the III century A.D. In any case the "Vatican" manuscript also contains several interpolations, as can be seen from a comparison with a few fragments found in papyri and with EUCLID's quotations occurring in various sources. Interpolated passages recognized as such by modern scholars are pointed out both in EUCLID, vols. I-V, and in HEATH whenever they occur. Interpolations usually consist in illustrative and explanatory additions.

Trying to put on a sound footing the issue of the authenticity of the first seven definitions[5] of the *Elements* we can in the main use three kinds of textual evidence: the one furnished by papyri, testimonies of authors who had had the opportunity of reading EUCLID's text in a form nearer to the original than the extant recensions, and the evidence given by the extant recensions themselves. Finally, one should not completely neglect later testimonies, since some of them might contain valuable information about more ancient traditions.

2. Evidence drawn from papyri

Unfortunately analysis of papyri can throw very little direct light on our issue, since only a few fragments of the text of the *Elements* have been found in papyri and only two such papyri contain definitions of Book I. One of them (P. MICH. III, 143, also in STAMATIS, vol. I, Appendix II, 187–188) contains the first 10 definitions of Book I, essentially in the same form contained in the manuscript tradition; the papyrus, however, dates back to the III century A.D., so is presumably roughly contemporary with the recension transmitted by the Vatican manuscript. The papyrus, probably written by a schoolboy, only contains the text of the definitions, lacking in particular any reference to EUCLID. Hence the papyrus only proves that at its time the first ten definitions were taught in the extant form, but it does not give any independent evidence for their attribution to EUCLID. The second papyrus (P. Hercul. n. 1061, described in HEIBERG, 1900, 161) contains only one definition and, coming from Herculaneum, it is certainly some centuries earlier than the presumed time of the corruption of the Euclidean text (that is the III century A. D. See below). The definition contained in this papyrus does not concern, however, an

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[3] Both Heiberg's conclusions have been contested by JONES (1994).

[4] Even in the case of the *Elements* Heiberg's attributions have been recently doubted by KNORR (1996), as A. J ONES has pointed out to me.

[5] Some of the subsequent definitions may raise, of course, analogous problems. The choice of explicitly considering here only the first seven definitions is mainly due to three reasons: their constituting a homogeneous block, whose common origin can hardly be doubted; the particular relevance of the geometrical entities therein defined; and the availability of a larger textual evidence.

elementary geometric object, but it is the following (unexceptionable) definition of a circle:

Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον, πρὸς ἥν ἀφ' ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι ἴσαι ἀλλήλαις εἰσίν.

A circle is a plane figure contained by one line [such that] *all the straight lines falling upon it from one point among those lying within the figure are equal to one another.*

The text transmitted by all known manuscripts of the *Elements* (definition I, 15) is the following:

Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον, ἥ καλεῖται περιφέρεια, πρὸς ἥν ἀφ' ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι πρὸς τὴν τοῦ κύκλου περιφέρειαν ἴσαι ἀλλήλαις εἰσίν.

A circle is a plane figure contained by one line, which is called the circumference, [such that] *all the straight lines falling upon it from one point among those lying within the figure to the circumference of the circle are equal to one another.*

A comparison between the two texts allows us to draw the following conclusions:

1. The original text of Book I of EUCLID's *Elements* did contain some of the definitions transmitted by the manuscript tradition, such as the one of circle.

2. EUCLID did not hesitate in using geometrical terms he had not defined in advance. The term περιφέρεια (*circumference* or, more generally, *boundary*), whose definition was missing in the original text of def. 15, is in fact used in the following definitions.

3. The use of geometrical terms not previously defined was avoided in the Imperial age, at least in some instances (as in the case of circumference), by inserting into the text of the *Elements* definitions originally missing.

The added phrase πρὸς τὴν τοῦ κύκλου περιφέρειαν (to the *circumference of the circle*) specifies again the second extremity of the straight lines, which in the original sentence had been clearly indicated by means of the words πρὸς ἥν; the original phrase is nevertheless also preserved, showing the dull zeal of the editor, who had tried to "complete" EUCLID's text by inserting any kind of terms he considered lacking, but had not dared omit any original word.

It is worth noting that, before the discovery of the Hercolanensis papyrus quoted above, the two interpolations had been recognized as such by HEIBERG, who had noticed that they were omitted by many ancient sources, such as PROCLUS, TAURUS, SEXTUS EMPIRICUS and BOETHIUS.

Our conclusions can be confirmed by several pieces of evidence[6].

[6] HEATH remarks that in some other instances, as in the case of *deflection* (κεκλάσθαι) and *verging* (νεύειν), EUCLID uses terms not previously defined by him, although the same terms had been defined by more ancient authors, as results from some passages from ARISTOTLE. Cp. HEATH, vol. I, p. 150. HEATH also remarks that *later the tendency was again in the opposite direction* (i.e. in the Imperial period more definitions were again included in textbooks).

3. A passage of Sextus Empiricus

SEXTUS EMPIRICUS is one of the main sources on Hellenistic thought. Since he writes before the extant recensions of EUCLID's treatise were edited and in his works he has many occasions to deal with the subject of definitions of geometrical objects, on our problem his testimony may be of great help. Our interest is increased by the circumstance that SEXTUS apparently could still read the original text of the *Elements*[7], as it results, in particular, from a comparison between some of his quotations and fragments of EUCLID's work found in papyri: in particular the definition of a circle quoted by SEXTUS EMPIRICUS (*Adv. Math*, III, 107) is the same definition contained in the Herculanensis papyrus.

SEXTUS EMPIRICUS' testimony at first sight seems to support the authenticity of the definitions we are here interested in. SEXTUS in fact repeatedly seems to quote definitions of fundamental geometrical entities contained in Book I of the *Elements* and in particular the definitions of a point, a line, a surface and a straight line. Let us examine first a passage in *Adversus Mathematicos* concerning the notion of point. It runs:

... *they* [the mathematicians], *in describing these* [the geometrical objects], *say that a 'point' is a 'sign' without parts or extension or the extremity of a line*...[8]

Since the sentence *the point is a 'sign' without parts* almost coincides with definition I, 1 of the *Elements* and the phrase about a point as *extremity of a line* with definition I, 3, EUCLID's work has been generally considered the source of the above passage. We can remark, however, that also a third property of the point is here mentioned, namely the absence of extension; furthermore none of the above properties is said to be a *definition* (ὅρος); SEXTUS quotes sentences which the *mathematicians* say *describing* (ὑπογράφοντες) geometrical entities, whereas when he had reported the definition of a circle (which we know, thanks to the Hercolanensis papyrus, to be, in all likelihood, the original one) SEXTUS EMPIRICUS had referred to mathematicians *who define* (ὁριζόμενοι) the circle (SEXTUS EMPIRICUS, *Adv. Math*, III, 107). Since in the *Elements* there are many *definitions*, but no *description*, we may suspect that SEXTUS EMPIRICUS is here referring not to EUCLID but to somebody else.

The identification of the mathematician actually referred to by SEXTUS is not too difficult, since the description of the point reported by SEXTUS is nothing but the initial part of the first of *Heron's Definitions* (HERON OF ALEXANDRIA, vol. IV)[9]. HERON, *describing* (ὑπογράφων), as he says, the geometric objects, had started with the following sentence:

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[7] HEIBERG (1888), on the ground of various elements, reaches the conclusion that SEXTUS EMPIRICUS could still read EUCLID's original text, whose corruption he dates to the third century A. D.

[8] ...ὑπογράφοντὲς λέγουσι στιγμὴν μὲν εἶναι σημεῖον ἀμερὲς καὶ ἀδιάστατον ἢ πέρας γραμμῆς ... (SEXTUS EMPIRICUS, *Adv. Math.*, III, 20). For the terms στιγμή and σημεῖον cp. below, Sect. 7.

[9] HERON's work was included in a Byzantine collection. The title *HERON's* Definitions was used by the Byzantine editor to distinguish the extant text (possibly extracted from a larger work) from extracts from other authors (cp. HEIBERG's introduction in HERON OF ALEXANDRIA, vol. IV, p. iv.); it seems unlikely that the title is the original one.

A point is that which has no parts or an extremity without extension or the extremity of a line. ...[10]

Heron's statement includes not only the two definitions which appear in the *Elements*, but also the two points which, as we had remarked above, make SEXTUS' quotation different from such definitions, namely the use of the verb to *describe* (ὑπογράφειν) and the characterization of a point as something without extension. SEXTUS actually repeats HERON's wording, except for two details: the replacement with the adjective ἀμερὲς of the equivalent expression οὗ μέρος οὐθὲν and one more πέρας in HERON's sentence (which, however, might well be a mistake of the copyist[11]).

Let us remark that scholars like HEIBERG and HEATH could not take into account HERON as a possible source of SEXTUS EMPIRICUS, since they, without the benefit of NEUGEBAUER's subsequent dating of HERON to the I century A.D.[12], had believed the III century A.D. the most likely date for HERON, who consequently should have lived after SEXTUS EMPIRICUS (active about 200 A. D.).

If the author referred to is HERON and not EUCLID, SEXTUS' passage becomes a significant clue against the authenticity of our definitions. SEXTUS EMPIRICUS is an exponent of scepticism, aiming at a radical criticism of the bases of mathematics. If at his time EUCLID's treatise did contain the first definitions included in our manuscripts, it would be difficult to understand why SEXTUS should have preferred to argue against the *descriptions* furnished by a popularizer of EUCLID's geometry, like HERON, instead of directly criticizing EUCLID's *definitions*.

4. The Definitions by Heron of Alexandria

The first seven definitions of the *Elements* all coincide with parts (in most cases the initial parts) of the corresponding "definitions" of HERON. Namely:

- *Elements*' def. 1 coincides with the initial part of HERON's def. 1.
- *Elements*' def. 2 coincides with the initial part of HERON's def. 2.
- *Elements*' def. 3 corresponds to another passage of HERON's def. 1.
- *Elements*' def. 4 (apart from the omission of the words μὲν οὖν, which in HERON have the function of a generic syntactic link with the context) coincides with the initial part of HERON's def. 4.

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10 σημεῖόν ἐστιν, οὗ μέρος οὐθὲν ἢ πέρας ἀδιάστατον ἢ πέρας γραμμῆς.

11 This statement is based on the following considerations: (1) the adjective ἀδιάστατον (*without extension*) may be referred directly to σημεῖον (*point*) and much better than to πὲρας (*extremity*); (2) repeating two consecutive times that a point is an extremity does not make much sense; (3) it is perhaps strange saying that a point is an extremity without specifying of what, but if one, for brevity, omits *of a line*, it surely does not make sense to add the complete sentence soon afterwards. The most likely possibility seems the one that the copyist, in copying the sentence quoted by SEXTUS, had omitted the words καὶ ἀδιάστατον and, having realized his mistake soon afterwards, had remedied this by producing the extant sentence.

12 NEUGEBAUER's dating rests on the identification of the lunar eclipse described by HERON in *Dioptra, 35* as the eclipse of A. D. 62, March 13.

- *Elements*' def. 5 (apart from the insertion of a δέ, with the function of a weak adversative link with the previous definition) coincides with the initial part of HERON's def. 8.
- *Elements*' def. 6 corresponds to another passage of HERON's def. 2.
- *Elements*' def. 7 coincides with the initial part of HERON's def. 9.

The coincidences are clearly too numerous to be casual, especially if we consider that the above are not true *definitions*, in the proper sense of the word, but vague sentences, built with non-technical language.

According to the usual interpretation, the coincidences occur because HERON's definitions are illustrative extended versions of EUCLID's definitions contained in the *Elements*. The above interpretation seems to be confirmed by the explicit reference of HERON to EUCLID in the introduction to his work (For HERON's passage see below, Sect.9).

SEXTUS EMPIRICUS' testimony examined in §3 suggests however another possibility. If SEXTUS could not draw his definitions from the *Elements*, but had to use HERON's work, the definitions included in the manuscript tradition of the *Elements* may be excerpts from HERON's writing, interpolated into EUCLID's treatise in late antiquity.

If the first seven definitions of the *Elements* are genuine, the explicit reference of HERON to EUCLID, implying a direct relation between the two texts, leaves no alternative to the first possibility. If instead the definitions are not original, their insertion in the *Elements* should have occurred not before the III century A. D., which is the generally accepted date of the corruption of EUCLID's text (cp. above, note 7). Hence they should have been missing in the text of the *Elements* used by HERON and should have been interpolated after centuries of the use of HERON's *Definitions* in teaching mathematics. In this second case a derivation from HERON is the only plausible explanation of the link between the two texts. The link is therefore in any case due to the derivation of one text from the other, whereas we can exclude different possibilities (such as a derivation of both texts from a common source). Hence any argument against the authenticity of the first seven definitions of the *Elements* shall also be an argument supporting the hypothesis of their derivation from HERON.

5. Plausibility of a derivation from Heron's Definitions

In order to show the plausibility of the conjecture proposed in the previous section, we have to answer the following two questions:

a) If HERON did not draw from EUCLID his definitions of fundamental geometric entities, what might have been his sources?
b) Through which channels might excerpts of HERON's definitions have been incorporated in the *Elements* ?

It is not too hard to answer the first question, since possible sources of HERON, alternative to EUCLID, are easy to find. Let us consider again, for example, HERON's *definition* 1. Its first part runs:

A point is that which has no parts, or an extremity without extension, or the extremity of a line, and, being both without parts and without extension, it can be grasped by the *understanding only. It is said to have the same character as the moment in time or the unit having position. It is the same as the unit in its fundamental nature, for they are both indivisible and incorporeal and without parts; but ... they differ...*[13].

Many of the ideas here reported by Heron can be found in Aristotle's writings. Among Aristotle's passages on this subject we may quote, e.g., *Phys.*, IV, 11, 220a 15 ff., where, among other things, points are said *extremities of lines* and the analogy between point and instant of time is introduced; *De Cael.*, III, 1, 300a 14, where the above analogy is also considered; *Met.*, V, 6, 1016b 24–30, where that which is indivisible and has position is called point. This definition of a point as a monad having position is in fact an ancient Pythagorean definition, as we know from Proclus (95, 22). As another example, we may recall that the definition of lines as extremities of surfaces, which is contained in a passage of Heron's definition 2 (corresponding to definition I, 6 of the *Elements*), is the Platonic definition which Aristotle had criticized in *Topica*, VI, 6, 143 b 11. In other instances Heron may have used later sources than Euclid: for example the definition of a straight line, as we shall see in detail in Sect. 10, seems to be drawn from Archimedes.

As concerns the channels through which excerpts from Heron's definitions may have been incorporated in the *Elements*, we have to remember that his *Definitions* are not the sole testimony to Heron's activity as a popularizer of geometry. We know in fact, from both Proclus (V century A. D.) and Arabic sources, that Heron had written a commentary on the *Elements*, aimed at a popularization of Euclid's work[14]. Many passages of Heron's commentary are quoted (apparently verbatim) by the Arabic scholar An-Nairīzī in his own commentary to Euclid. The extant *Definitions* may perhaps be excerpts from the commentary, but even if they are a different work, passages from Heron's *Definitions* might have been inserted in the commentary to the *Elements* either by Heron or by later editors. In this case the distinction between such passages and Euclid's text should have had a very little chance of being preserved in later centuries, when Euclid's treatise was copied for mere didactical purposes, without any philological care.

Some passages contained in all known manuscripts of the *Elements* have been identified as interpolations coming from Heron's commentary on the basis of various testimonies. Some such passages have been attributed to Heron on the basis of Heron's excerpts reported by An-Nairīzī. (for instance proposition 12 of Book III; cp. Heath, vol. 2, pp. 28–29). Other passages are attributed to Heron by Proclus but not by An-Nairīzī (for instance an alternative proof of prop. I, 25; cp. Proclus, 346–347). Since, furthermore, some passages of Book I of the *Elements* recognized as authentic by Proclus have been proved to be an interpolation[15], we have to infer that neither Proclus nor An-Nairīzī can warrant the authenticity of the text they consider original. Evidently both

[13] I have used the English translation by Ivor Thomas.

[14] The Arabic glossary *Fihrist,* s.v. *Heron,* says that *He wrote the book of explanations of the obscurities in Euclid* (see Suter, 16. I have used the English translation in Heath, vol. I, p. 21).

[15] The evidence given by papyri proves that Proclus only knew a corrupted text of the *Elements*. Proposition 40 of Book I, which is found in all the manuscripts and is recognized by Proclus, is in fact missing in a papyrus, which also gives a better text of prop. I, 39 (cp. P. fayûm 9, also

PROCLUS and AN-NAIRĪZĪ used manuscripts in which the original Euclidean text was not distinguishable from later additions, coming in particular from HERON's commentary, and their attributions to HERON were grounded on other sources (probably ancient commentaries), afterwards lost. The scant reliability of PROCLUS' attributions to EUCLID may also be illustrated by his recognizing as authentic two pseudo-Euclidean works, namely the *Catoptrics* and the *Elements of Music*, which are certainly spurious (cp. PROCLUS, 69).

To sum up, we can answer the above questions by stating that:

a) Even leaving out of consideration EUCLID's work, HERON should have had at his disposal sufficient material for drawing up his "definitions" of fundamental geometrical entities.

b) Since no extant recension of the *Elements* can be dated before the III century A. D., and scholars of the II and III century used EUCLID's text together with HERON's commentary, and since at that time the editors of the *Elements* certainly had no philological care, the insertion of excerpts drawn from HERON in the text of the *Elements* appears quite plausible, and in some instances is also directly documented.

It is conceivable that a list of HERON's definitions was truncated in order to get a set of short "definitions" suitable to be learnt by heart in the schools: the papyrus quoted in Sect. 2, dated to the III century A.D., might have been a late specimen of a list of this kind. If such a list was usually premised to the *Elements*, it could hardly avoid being eventually confused with EUCLID's text.

6. Heron's version of two definitions of Book I of the *Elements*

In the present section we shall compare two definitions of Book I of the *Elements*, namely def. 15 (of circle) and def. 22 (of various quadrilaterals), with the corresponding definitions given by HERON.

The definition of a circle given by HERON (*Definitions*, def. 27) begins as follows:

Κύκλος ἐστὶ τὸ ὑπὸ μιᾶς γραμμῆς περιεχόμενον ἐπίπεδον. τὸ μὲν οὖν σχῆμα καλεῖται κύκλος, ἡ δὲ περιέχουσα γραμμὴ αὐτὸ περιφὲρεια, πρὸς ἥν ἀφ' ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι ἴσαι ἀλλήλαις εἰσίν.

A circle is a plane figure contained by one line. The figure is called circle, and the containing line circumference, if all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

A comparison of this with the two definitions of a circle reported above in Sect.2 shows that HERON had included one of the two interpolations present in the manuscript tradition of the *Elements*. This circumstance suggests the possibility that HERON's definition may be representative of an intermediate stage in the corruption of EUCLID's text.

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in STAMATIS, vol. I, Appendix II pp. 188–189). AN-NAIRĪZĪ was not, in all likelihood, in a better position than PROCLUS.

In any case the almost complete coincidence with the original definition documented by the Herculanensis papyrus allows us to conclude that HERON had drawn his definition from EUCLID.

Let us read now def. I, 22 of the *Elements*. It runs:

Τῶν δὲ τετραπλεύρων σχημάτων τετράγωνον μέν ἐστιν, ὃ ἰσόπλευρόν τέ ἐστι καὶ ὀρθογώνιον, ἑτερόμηκες δέ, ὃ ὀρθογώνιον μέν, οὐκ ἰσόπλευρον δέ, ῥόμβος δέ, ὅ ἰσόπλευρον μέν, οὐκ ὀρθογώνιον δέ, ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευράς τε καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οὔτε ἰσόπλευρόν ἐστιν οὔτε ὀρθογώνιον· τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλείσθω.

Of quadrilateral figures, a square is that which is both equilateral and right-angled, an 'eteromekes' that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

The initial sentence is of course a correct definition of a square. Since squares are used in fundamental propositions of the Elements, such as the so-called *Pythagoras' theorem*, there is no reason for doubting the authenticity of the definition. The following sentences may be suspected to be an interpolation, because *eteromekes* (the non-square rectangle), rhombus, rhomboid and trapezium are figures that are never used in the *Elements*. All modern scholars agree that their definitions were drawn from older textbooks and inserted for sake of completeness. It may be doubted whether the author of the insertion was EUCLID, as generally supposed (cp. e.g. HEATH, vol.1, p. 62), or some later editor. Some evidence on the definitions of a rhomboid and a rhombus is given by GALEN[16].

The conclusion of the definition looks particularly strange, since the term τραπέζιον (trapezium) is used for a generic quadrilateral, without parallel sides. The property of having two parallel sides had presumably characterized the meaning of the term τραπέζιον (a diminutive of τράπεζα, which means "board", in particular a board used as dining table) since its first use in geometry. As a matter of fact the term is used in our meaning of *trapezium* by EUCLID himself (in his treatise *On division of figures*) and in later literature: in particular by ARCHIMEDES[17], STRABO (*Geography*, II, 5, 33),

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16 GALEN, *In Hippocratis librum de articulis et Galeni in eum commentari*, vol. 18a, 466, 15. GALEN, after having defined *rhomboids* as equilateral but not rectangular figures, adds the remark "so indeed EUCLID defines a rhombus". GALEN uses the word *rhomboid* (ῥομβοειδές) in the meaning of *rhombus-shaped*: a traditional use of the Greek word, but one which is inconsistent with the definition appearing in the *Elements*. This discrepancy (which is very suspicious in a sentence explicitly referring to EUCLID) might have caused the later insertion of the remark. Unless, of course, in the *Elements* known to GALEN only the figure of rhombus, but not the one of rhomboid, was defined. In both cases GALEN's testimony seems to give a clue against the genuineness of our text of definition 22.

17 ARCHIMEDES systematically uses the term τραπέζιον for a quadrilateral having two parallel sides. Cp., e.g., *On the Sphere and Cylinder*, 28, 5; 35, 12; *Quadrature of the Parabola*, 176 ff. In only one place (*On Plane Equilibriums*, 99, 6–7) the word trapezium (τραπέζιον) is followed in our manuscripts by the specification "having two parallel sides". Before assuming an incoherent use of terminology by ARCHIMEDES, one must rather suppose in this case an interpolation due to

POSIDONIUS (in his classification of quadrilaterals referred to by PROCLUS, 170–171) and, as we shall see in a moment, HERON .

HERON 's *definitions* from 51 to 63 concern quadrilaterals. They almost coincide with the sentences included in def. 22 of the *Elements,* except for some important differences.

The first difference is the insertion, between rhomboids and trapezia, of the definitions of a parallelogram and a gnomon. These definitions are based on the notion of parallelism, which, strangely enough, has not yet been introduced[18]. The first sentence of HERON not using explicitly such a notion is def. 59. It runs:

Τῶν παρὰ τὰ εἰρημένα τετραπλεύρων ἃ μὲν τραπέζια λέγεται, ἃ δὲ τραπεζοιδῆ .

Of quadrilaterals other than these some are called trapezia and others trapezoids.

Afterwards, in definitions 60 and 61, the two cases are distinguished depending whether the quadrilaterals have two parallel sides or not.

If, as we have conjectured, an editor had decided to extract from HERON's work a short list of geometric definitions, he might have preferred to avoid the notion, not yet introduced, of parallelism; even in this case he could have hardly missed a transcription of HERON 's def. 59, which appears a natural conclusion of the classification of quadrilaterals and does not contain any explicit reference to the notion of parallelism. In absence of such a notion, however, our editor could not follow HERON in his distinction between trapezia and trapezoids, but had to use a single term for both cases. Since the two terms present in the source were trapezium and trapezoid, the use of the term *trapezium* (τραπέζιον) for the generic case was certainly the most obvious choice.

To sum up, definition 22 of the *Elements* can be obtained by adding to the definition of a square the transcription of all HERON's definitions which concern quadrilaterals and do not use the term "parallel", introduced in the next definition 23[19]. The final part of definition 22 is so easily understandable as being derived from HERON and so difficult to explain otherwise that we are left with very few doubts about its origin. We can reasonably infer that contaminations between HERON's and EUCLID's works have occurred not only in the case of some propositions (as we have seen at the end of Section 5), but also for some definitions of Book I.

The comparison between definitions 15 and 22 and the corresponding definitions in the Herculanensis papyrus and in HERON's work allows us to draw the following conclusions:

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the influence of the terminology used in the transmitted recension of the *Elements*. Likewise, APOLLONIUS' terminology on conics (not yet introduced at ARCHIMEDES' time) is used in a couple of passages in our manuscripts of ARCHIMEDES' works.

[18] Parallel lines are only introduced in HERON's definition 70 (and in our *Elements* in def. 23). HERON's order may be explained by supposing that HERON, supplementing EUCLID's text with new definitions (in particular of parallelograms and gnomons), had not altered the order of original EUCLID's definitions.

[19] It is worth noting that the imperfection of the classification of quadrilaterals given in def. 22 is attributed by PROCLUS too (l.c.) to the circumstance that the notion of *parallel lines*, not having been yet introduced, could not be used.

1. HERON's *Definitions* contain not only complemented and illustrated versions of EUCLID's definitions (as in the case of the circle) but also definitions of geometrical entities which EUCLID had not defined at all, such as circumference and trapezium. Some of these entities (not all of them) are nevertheless defined in the manuscript tradition of the *Elements*: this is the case of the circumference.

2. Some of the definitions missing in the original text of EUCLID and later included in Book I of the *Elements* were drawn from HERON's work. This was very likely the case of trapezium and perhaps also of the various quadrilaterals other than squares introduced in definition 22 (and maybe of the circumference too).

7. Some methodological considerations

According to the previous section both possibilities considered in Sect. 4 seem to have actually occurred for some of definitions of Book I: definition 15 (if the interpolated definition of a circumference is left out of consideration) seems to have moved from the *Elements* to HERON's *Definitions*, whereas both movements (first from the *Elements* to HERON and then the other way round) seem to have occurred for the content of definition 22.

The first seven definitions of Book I constitute, from all points of view, an homogeneous set and we may therefore suppose a common origin for them. The main criterion for determining this is the internal consistency of the work. The *Elements*, and its Book I in a particular way, have an unitary structure, all propositions being linked to each other by means of relations of strict logical implication. The conjecture that the definitions here considered are an interpolation is therefore suggested, apart from the textual evidence discussed in the previous sections, by the fact that not only are they never used in EUCLID's work, but it would be quite impossible to use them. Their elimination would result in a strengthening of the methodological consistency of the *Elements*.

In the history of thought we can recognize two completely different views about the function of definitions. According to the first view definitions have the purpose of describing the true essence of the defined entities, whose actual existence is of course considered to be independent of their definitions. In the case of mathematics this concept, which was sustained by PLATO and was dominant from the Imperial period until (at least) the XVIII century, implies, of course, that mathematical entities actually exist, independently from the mathematicians describing and using them. We shall call this first view *essentialist* or *Platonic*. This view was substantially shared by ARISTOTLE[20].

[20] ARISTOTLE had at length criticized PLATO's view of an independent existence of mathematical objects (cp. *Met.*, XI, 4; XIII; XIV), maintaining that they exist only as properties of sensible bodies. Even though the philosophical grounds of ARISTOTLE's and PLATO's concepts may appear at first sight quite different, the difference does not change in relevant measure the mathematician's attitude toward his own job; in this respect (which is the one we are here interested in) the point is that for both ARISTOTLE and PLATO men do not construct mathematical entities, which exist, in some sense, independently of them. For ARISTOTLE's view that defining something amounts to describing its essence, cp., for instance, *Topica*, I, 5, 101b 36; *Met.*, 1042a 17; 1031a 13. On substantial Platonism of ARISTOTLE's views on definitions cp. POPPER, vol. II, pp. 10–11.

According to the second view, which we shall call nominalist, the function of definitions is the introduction of a short label for a long defining formula. The existence of the object so defined must of course be ascertained by other means. In the case of mathematics the existence of the defined objects can be warranted by means of their actual geometrical construction. This second concept, so completed, shall be called *nominalist* and *constructivist*[21]. It implies that mathematical objects are conceptual tools built by men by means of definitions. People sharing this second concept certainly realize the necessity of avoiding a regress 'ad infinitum' by assuming some undefined terms as a starting point.

It is evident that the two above views are incompatible with each other and that the seven definitions here considered make sense only for people sharing the *essentialist* or *Platonic* concept. Our problem is therefore reduced to the one of finding EUCLID's own view. To this purpose the following considerations may be helpful.

First we remark that the second opinion, and in particular the admission of the necessity of leaving some terms undefined, is not a modern view, as its long oblivion might suggest, but it is attested before EUCLID. ARISTOTLE mentions, in fact, that according to ANTISTHENES' school it is possible to give a definition of the composite kind of things or substances, whether they are sensible things or objects of intellectual intuition, but not of their primary parts (ARISTOTLE, *Met.*, 1043b, 23–32).

The choice of avoiding a regress 'ad infinitum' by basing mathematical definitions on some undefined elementary terms is quite analogous to the one of founding demonstrative chains on indemonstrable postulates. EUCLID is, to our knowledge, the author of the first work in which demonstrations are explicitly based on indemonstrable postulates. Why should not EUCLID himself have realized the necessity of basing his definitions on undefined elementary terms? It is true that in medieval and modern mathematical thought the understanding of the necessity of basing definitions on undefined terms was less diffused by far than the one of founding demonstrations on indemonstrable postulates, but in that case the difference may just be due to the overwhelming influence of the transmitted redaction of the *Elements*[22].

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[21] The important point that in Hellenistic mathematics the existence of geometric objects was warranted by their construction was first stressed by ZEUTHEN. For a different view see KNORR (1983). In any case it would be very interesting, in my opinion, to investigate the possible influence of historical studies like those by ZEUTHEN on the rising of modern constructivism. We should remember that one century ago, unlike today, historical studies on Greek mathematics did not constitute a specialistic field with a very narrow audience, but were of great interest to most mathematicians.

[22] One might ask why compilers of imperial age should have accepted undemonstrable postulates, but not undefined terms. The answer is simple enough: pre-Euclidean tradition could furnish many Platonic definitions of elementary geometrical objects, but no proof, to be sure, of EUCLID's postulates. Many attempts of proving Euclidean postulates are however well documented in the Imperial age in the case of the fifth postulate. One might also speculate about the possibility that in some instances such an attempt, considered successful, could have changed one of EUCLID's postulates in a proposition of the extant *Elements*. It is conceivable that the above description might explain the origin of proposition I, 4. A few words on ARISTOTLE's views are needed here, since it seems that he admits the necessity of first principles as starting points of demonstrative arguments, but he never says anything about undefined terms. The *first principles* ARISTOTLE talks

The view which we have called *nominalist* and *constructivist* is apparent in other definitions, certainly genuine, contained in the *Elements* and in particular in the one of proportion. If, in fact, one conceives the "ratios between magnitudes" as something actually existent in nature, the equality of two ratios appears an obvious notion, whereas Euclid adopted a definition equivalent to a complex construction of the notion of ratio between magnitudes[23].

The previous remark is consistent with the circumstance that all authors who have shared Platonic views have accepted, without raising any objection, the definitions inserted at the beginning of the *Elements*; at the same time they could not understand the utility of Euclid's complex definition of a proportion[24], preferring also in this case (as, for instance, Galileo did) a Platonic definition. The seven definitions we are here concerned with were accepted without problem for many centuries and only started to be criticized when the capacity of understanding definitions like the one of proportion was recovered.

Without entering into a thorough analysis of the *Elements* (which would give, I believe, other proofs of its constructivist nature), let us consider an example relative to one of the geometrical entities with which our definitions are concerned: the point. The idea of point (στιγμή) had been at length analysed, before Euclid, in the framework of Platonic views. Also, for instance, the analyses of point contained in Aristotle's passages quoted above (in Sect. 5) had been of this kind. Euclid makes clear how remote his views are from this concept by avoiding, in his treatise, the term στιγμή, replaced by him with the new term σημεῖον, i.e. *sign*, or *mark*[25]. The first postulate, for instance, which nowadays is usually given by stating the existence of a straight line passing through any two given points, in Euclid's text was:

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about have however little to do with Euclid's postulates. It rather seems that he has in mind a kind of essentialist definitions (cp., in particular, *Anal. Post.*, I, 10, 76a, 40, where Aristotle, as examples of first principles of geometry, considers statements about what a line actually is). In Aristotle's view such *principles* are used as grounds for both definitions and theorems.

[23] Euclid's definition is substantially equivalent, as many scholars have remarked, to the modern definition of real number, which actually is a translation of the ancient definition in modern language. There is however an important difference, since Euclid, unlike modern mathematicians, only takes into account ratios between magnitudes whose existence have been already proved by means of actual geometrical construction.

[24] This definition is usually attributed to Eudoxus. Without discussing here the grounds for such an attribution, I only remark that, since the authenticity of the definition has never been doubted, even if Euclid should have simply transcribed Eudoxus' definition, its insertion in the *Elements* would anyhow be valuable evidence on Euclid's views on definitions.

[25] The new term was apparently introduced by Euclid. The occasional presence of the word σημεῖον in a few passages of Aristotle is irrelevant, since the passages belong to works, if not apocryphal, at least known to us in recensions probably later than Euclid. The term appears, in particular, in some geometrical constructions contained in the *Meteorologica*. Aristotelean authorship of Meteorologica, IV was successfully contested by Gottschalk, who gave good reasons for associating it with Theophrastus. A composite authorship of Meteorologica, III (where the term σημεῖον repeatedly appears in the meaning of 'point') was suggested by Jones (1994). In any case the influence of Euclid's terminology on the subject of geometrical constructions obviously did reach even to manuscript tradition of classical works.

Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν, i.e., literally:

Let the following be demanded: to draw a straight line from any 'sign' to any 'sign'.

One could hardly image a clearer way to break away from all tradition of Platonic speculations on geometrical entities and to emphasize that mathematics is not a description of actually existing objects, but a model of particular human activities, in this case of drawing. The widespread belief that EUCLID had produced a Platonic definition of a '*sign*', such as the one included as definition 1 in Book I of the *Elements*, appears quite inconsistent with his clear terminological choice. It is worth noting that, whereas in the early Hellenistic period (in particular in ARCHIMEDES' and APOLLONIUS' works) only the Euclidean term for 'point' was used, in the Imperial age also the word στιγμή (which we have found in SEXTUS EMPIRICUS' writing) came again into use; EUCLID's term was eventually completely replaced by the older στιγμή .The latin word *punctum* (from which most of modern European terms for 'point' are derived) is in fact a literal translation not of EUCLID's term, but of the one used by PLATO and ARISTOTLE (the term στιγμή means *puncture*). The return of pre-Hellenistic concepts had evidently extended its influence also to mathematical terminology.

There is a close relation between the view which we have called *nominalist* and *constructivist* on the problem of definitions and linguistic conventionalism. It is significant, in our concern, that PLATO and ARISTOTLE had never considered the possibility of enriching the language with the introduction of new conventional terms. Such a possibility was instead systematically exploited in the early Hellenistic period both in anatomy (in particular by HEROPHILUS OF CHALCEDON, see H. VON STADEN) and in mathematics (for instance by ARCHIMEDES and by APOLLONIUS of PERGA[26]) and it was again given up in the Imperial period, when Platonic views were re-established on the problem of definitions too. The example of the new term σημεῖον introduced by EUCLID for "point" (which we have examined in the previous paragraph) suggests that EUCLID had shared, on the subject of language, the conventionalist opinion of his contemporary HEROPHILUS and of later Hellenistic mathematicians[27].

The *Elements*, being an elementary handbook copied for didactical purposes, had little chance of remaining unaltered. Hence our understanding of mathematical methodology in the early Hellenistic period rests particularly on the analysis of works, belonging to the same scientific tradition, which have undergone fewer alterations thanks to their nature of more advanced texts, chiefly ARCHIMEDES' and APOLLONIUS' writings. In these works there is nothing analogous to the pseudo-definitions of fundamental geometrical

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[26] Cp., for instance, the new terminology introduced by APOLLONIUS OF PERGA in his *Conics* and the terminology introduced by ARCHIMEDES in his treatise *On Conoids and Spheroids*.

[27] I think that there is also some methodological affinity between linguistic conventionalism, nominalistic theory of definitions and a relativistic theory of motion. Hence some indirect light upon our issue can be thrown by the remark that both HEROPHILUS and EUCLID opposed the Aristotelian theory about space and motion, holding that only relative motions can be observed (cfr. EUCLID's *Optics*, prop. 51 and, for HEROPHILUS, test. 59a in VON STADEN). Aristotelian theories on motion were re-established in the Imperial period together with Aristotelian views about definitions and language.

entities contained in the *Elements*. The introduction of terms implicitly defined through postulates is instead frequent, in particular in ARCHIMEDES' works[28]. Furthermore we know from PROCLUS that APOLLONIUS OF PERGA had discussed fundamental geometrical entities describing their genesis from everyday experience. In particular he had explained how the idea of line is generated by the consideration of things, like roads or walls, of which it is possible to ask somebody to measure the *length* without raising any doubt about the meaning of the request[29]. This kind of considerations appear particularly modern simply because of their distance from Platonic views. We know, on the other hand, that APOLLONIUS was faithful to Euclidean tradition and we do not know that he had criticized EUCLID's definitions[30].

An interesting, even though indirect, testimony is given by IAMBLICHUS. At the beginning of Book VII of the *Elements* an unity is defined as *that by virtue of which each of the things that exist is called one*. Such a *definition* (which clearly appears Platonic in nature and therefore analogous to the geometrical definitions which we are here discussing) is quoted by Iamblichus, who attributes it to *more recent* writers (οἱ νεώτεροι) (see IAMBLICHUS, 11). Even though the previously quoted mathematicians are Pythagoreans presumably more ancient than EUCLID, IAMBLICHUS (who often explicitly quotes EUCLID and in particular the arithmetical books of the *Elements*) would hardly have used this expression when referring to EUCLID. The omission of EUCLID in the long list of mathematicians whose definitions of an unity are reported by IAMBLICHUS strongly suggests that there was no definition of an unity in the *Elements* known to IAMBLICHUS.

This last testimony is consistent with the hypothesis that *more recent* authors might have also inserted in the *Elements* ancient Platonic definitions concerning geometrical terms.

The above considerations suggest that EUCLID did not share the Platonic concept which is the ground of the definitions we are here considering. Of course the presence of such definitions in a work which appears otherwise based on different concepts does not necessarily imply that they are non-genuine, since an eclectic attitude of EUCLID might well explain the apparent contradiction. We have to remark, however, that whereas such an eclecticism is not otherwise documented either for EUCLID nor for III century B. C.

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[28] I believe that the method of implicit definition is used, for instance, in the treatise *On Plane Equilibriums* for the notion of *centre of gravity* and in Book I of the treatise *On the Sphere and Cylinder* for the notion of *length* of a class of curves. The same method seems to be used by EUCLID for the term ἴσος (*equal*), which assumes the meaning of *equal in content* through the use of the common notions (this is also HEATH's opinion; cp. HEATH, vol. I, p.327). Of course if the reconstruction here proposed is accepted one has to conclude that also objects like point or straight line were implicitly defined in the *Elements* by the postulates. This, however, does not imply attributing to EUCLID a formalist view (see next footnote).

[29] Cp. PROCLUS, 100. APOLLONIUS' view referred to by PROCLUS does not hint to a formalist position. People thinking that objects like points or straight lines are the result of an abstraction process starting with very concrete objects may conclude rather that this origin must be taken into account in the choice of useful theorems and as a hint in their proofs, even though a description of the process of abstraction cannot lead to a formal definition.

[30] Unless, of course, one deduces (as some people did) APOLLONIUS' criticism toward EUCLID merely from his considerations about the origin of the idea of line.

mathematicians, it is quite consistent with the common attitude in the Imperial age, with the then renewed interest in ARISTOTLE and PLATO and, in particular, with the eclectic scientific attitude of HERON OF ALEXANDRIA, who did not hesitate in mixing heterogeneous cultural traditions (such as, for instance, Mesopotamian and Greek mathematics). The most likely conjecture is, therefore, the one that the contamination between the axiomatic-deductive structure of the *Elements* and "Platonic" definitions, which is apparent in HERON 's work, actually goes back to him.

We may conclude that, whereas the two possibilities considered above, in Sect.4, both appear compatible with our information on textual tradition (as it is proved by the fact that both occurred for other definitions of Book I), the thesis that the first seven definitions of the *Elements* are an interpolation drawn from HERON's work is more consistent with our knowledge of the history of Hellenistic scientific methodology.

It is worth noting that our conclusion seems to have been already reached, by implication, by KARL POPPER (vol. II, 9), when he writes:

The development of thought since Aristotle could, I think, be summed up by saying that every discipline, as long as it used the Aristotelian method of definition, has remained arrested in a state of empty verbiage and barren scholasticism, and that the degree to which the various sciences have been able to make any progress depended on the degree to which they have been able to get rid of this essentialist method.

If we apply POPPER's consideration to geometry and we exclude, as we must, that EUCLID's method (which has been the very model of scientific method for more than two thousands years) may be described as *empty verbiage and barren scholasticism*, we have to deduce that the definitions here concerned (which are not only Platonic-Aristotelian in nature, but often repropose verbatim PLATO's or ARISTOTLE's definitions; cp. above, Sect. 5) cannot be genuine. Strangely enough, POPPER did not draw the above conclusion from his sharp consideration, but held the traditional idea of a *Platonist* EUCLID[31].

The widespread belief in EUCLID's Platonism mainly originated, in my opinion, from the attempts of Platonist philosophers to trace back to PLATO (whose interest in geometry is evident) later scientific developments. The apparent success of such attempts may be due mainly to three reasons: the circumstance (far from being casual) that the only extant Greek commentary on EUCLID is that of the neoplatonist philosopher PROCLUS; the presence in our text of the *Elements* of the definitions which we are here discussing, which are clearly Platonic in nature; and the credit of Platonic interpretations to EUCLID, coming from the strength of Platonic views in almost all mathematical schools from the Imperial age to our times.

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[31] Cp. POPPER, vol. I, Addendum I. POPPER (319) also states that *Euclid's Elements are not a textbook of geometry, but rather the final attempt of the Platonic School to resolve this crisis* [i. e. the crisis due to the discovery of the irrationality of the square root of two] *by reconstructing the whole of mathematics and cosmology on a geometrical basis,...* Since the scientific relevance of the *Elements* cannot obviously escape him, POPPER has to conclude that PLATO was the *founder of modern science* . Since PLATO also was (in POPPER's opinion, too) the founder, or at least the inspiration for the Aristotelian essentialist method of definition, the last statement is hardly consistent with POPPER's passage quoted in the text.

8. Sextus Empiricus again

After the passage quoted in Sect. 3, concerning the notion of point, Sextus Empiricus goes on (*Adv. Math.*, III, 20):

..., *the line is length without breadth or the extremity of a surface, and the surface the extremity of a body or breadth without depth*.

Whereas the two definitions of a line and the second definition of a surface are present in both the *Elements* and Heron's work, the definition of a surface as extremity of a body is in Heron but is not included in the *Elements*. In his discussion concerning the straight line, Sextus Empiricus (*Adv. Math.*, III, 94, 96) quotes not only the definition included in both *Elements* and Heron's *Definitions*[32] but also another definition (grounded on the invariance of the line with respect to rotations leaving fixed two of its points), which is lacking in the *Elements* but is included in Heron's definition 4 (Sextus Empiricus, *Adv. Math.*, III, 98). In another place Sextus Empiricus also reports a definition of a line as what is produced by the flux of a point (Sextus Empiricus, *Adv. Phys.*, I, 376): another definition which does not appear in the *Elements* but does in Heron's work.

On the subject of geometrical definitions Sextus Empiricus certainly uses other sources besides Heron: for instance in the cases of angle[33] and circumference (where the source, as we have seen in Sect. 6, seems to be Euclid). When he reports sentences which appear in the same form in both the *Elements* and Heron's *Definitions* we cannot know whether he is referring to Euclid, to Heron or to somebody else (possibly to Heron's sources or to Heron's epitomizers). Since however in some instances (as in the case of the definition of a point examined in Sect. 3) it is possible to recognize Heron as the source, we certainly cannot use these passages as a proof that the definitions reported by Sextus were already present at his time in the *Elements*.

Particularly relevant to our concern is a passage of Sextus Empiricus (*Outlines of Pyrrhonism*, II, xvi, 207–208) in which he criticizes the use of definitions (ὅροι) in mathematics. In the English translation by R. G. Bury it runs:

And since, if we propose to define absolutely all things, we shall define nothing, because of the regress 'ad infinitum'; while if we allow that some things are apprehended even without definitions, we are declaring that definitions are not necessary for apprehending ... then we shall either define absolutely nothing or we shall declare that definitions are not necessary.

First of all the above passage shows that the possibility of building definitions starting from undefined terms (a possibility which, as we have seen in Sect. 7, seems to go back to Antisthenes' school) was still considered in Sextus Empiricus' time (about 200 A. D.). Secondly, Sextus Empiricus would hardly have criticized the definitions (ὅροι) given

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[32] The sentence reported by Sextus (who calls it a *description* and not a *definition* of straight line) differs slightly from the one in the *Elements*, the word *points* (σημείοις) being replaced by parts (μέρεσι).

[33] The origin of the *description* of angle referred to by Sextus E empiricus (*Adv. Math.*, III, 100) is unclear.

by mathematicians without taking into account EUCLID's *Elements*, i.e. the work which was the basis of all later mathematical developments (and a work very well known to him, as he repeatedly shows). Hence EUCLID's methodological choice on the subject of definitions must be among the possibilities taken into account by SEXTUS. Since EUCLID had neither avoided giving definitions, nor, of course, had listed infinitely many of them, the above passage suggests that in the edition of the *Elements* known to SEXTUS EUCLID had *allowed that some things were apprehended even without definitions,* i.e. that this edition did not contain definitions of fundamental geometrical entities. If this is actually the case, we also realize why on this subject SEXTUS EMPIRICUS had to use as his sources other authors, like HERON, EUCLID.

9. The testimony of Heron

We can draw some further elements supporting our thesis from a comparison between the first seven definitions of the *Elements* and the corresponding passages in HERON's work.

1. An important testimony is produced by HERON at the beginning of his *Definitions* (HERON OF ALEXANDRIA, vol. IV, 14):

In describing [ὑπογράφων] and sketching for you as briefly as possible, o *most excellent Dionysius, the technical terms premised in the elements of geometry [τὰ πρὸ τῆς γεωμετρικῆς στοιχειώσεως τεχνολογούμενα], I shall take as my starting point, and shall base my whole arrangement upon, the teaching of Euclid, the writer of the* Elements *of theoretical geometry;. . .*

This introduction gives an important support to our thesis, for the following reasons.

a) HERON, who refers to the *Elements* as the base of his work, says that he is illustrating not the geometrical terms defined in the *Elements*, but τὰ πρὸ τῆς γεωμετρικῆς στοιχειώσεως τεχνολογούμενα. HERON's sentence is consistent with the hypothesis that he had purposed to start his work by illustrating the fundamental geometrical entities left undefined in the *Elements,* which could be referred to just as τὰ πρὸ τῆς γεωμετρικῆς στοιχεώσεως τεχνολογούμενα (the *technical terms premised in the elements of geometry*). The possibility of interpretating the Greek phrase as referring to the *first technical terms contained in the elements of geometry* can be discharged on the basis of the consideration that many of HERON 's definitions, starting with some of the first few, are not included in the *Elements*: HERON 's definition 3, e. g., (where different kinds of lines are introduced) has no correspondence in the *Elements*.

b) HERON does not pretend to give 'definitions' of geometrical objects. In his own words, in fact, HERON is *describing* (ὑπογράφων) the geometrical technical terms and many of his *Definitions* (and in particular the first few of them) actually are long illustrations and not definitions. We can conclude that the difference between 'definitions' and 'descriptions', such as the ones which he furnishes of fundamental geometrical entities, was clear to HERON[34].

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[34] The title of HERON's work may appear to contradict this, but see above, note 9. The term ὅρος is never used by HERON in the body of the work in the sense of *definition*.

c) Finally we remark that the circumstance that HERON considers his exposition an useful preliminary to the reading of the *Elements* makes the conjecture particularly likely that either he himself or later editors had premised extracts of HERON's *Definitions* to EUCLID's treatise.

2. Another clue is given by the literal identity between the first seven definitions of the *Elements* and the corresponding passages in HERON. Such an identity may be considered a natural consequence of "scissors and paste" editing work, but it is hardly consistent with HERON's program of describing geometrical terms. We may suppose, of course, that HERON in the case of the first definitions had preferred to quote EUCLID verbatim. In this case we should explain, however, why HERON should have mixed in the same sentences EUCLID's quotations and his comments. We cannot exclude, of course, that HERON had constructed long sentences in his fluent style by expanding EUCLID's definitions, kept unaltered and used as prefabricated syntactic elements. The converse procedure, however, consisting in isolating simple propositions from the more complex sentences in HERON, may explain in a much more natural way the relation between the two extant texts.

3. A further element is given by the suspicious circumstance that point and line are both defined twice (point in definitions 1 and 3 and line in definitions 2 and 6). The insertion of two independent definitions of the same term is an evident logical incongruity and it is strange that such an incongruity could have escaped EUCLID. When in the *Elements* a proposition is demonstrated twice it has been always possible to prove the spuriousness of at least one of the demonstrations. These duplicated definitions can be easily explained as being derived from HERON, who had reported many different characterizations of point and line. If the compiler of the list of definitions afterwards included in the *Elements* had to decide which of HERON's sentences to keep as "definitions" to insert in the text, it is understandable that sometimes the choice was not easy and to keep two of them could appear the best decision. Let us illustrate the situation in the particular case of the point. In order to draw from HERON's long description (partly reported above in Sect. 5) a short "definition" of point, the most obvious device would have been the one of truncating HERON's passage, transcribing only his first proposition. The first five words of HERON's passage, σημεῖόν ἐστιν, οὗ μέρος οὐθέν (*a point is that which has no part*) actually constitute definition 1 of the *Elements*. It should have been very tempting, however, to retain some of the other characterizations of point too and in particular the one of points as extremities of lines. This second characterization is also included in the *Elements*, as definition 3. The two definitions of a line follow exactly the same pattern.

10. The definition of a straight line

A comparison between definition 4 (of a straight line) of Book I of the *Elements* and the corresponding definition by HERON furnishes an element which I consider decisive in support of the thesis here proposed. The definition included in the *Elements* is: εὐθεῖα γραμμή ἐστιν ἥτις ἐξ ἴσου τοῖςἐφ' ἑαυτῆς σημείοις κεῖται . An attempt at literal translation was given in Sect. 1: *a straight line is* [a line] *which lies uniformly in respect to* [all] *its points*.

In any case the meaning is obscure. A possible interpretation seems to be that the straight line has the property that there are rigid motions which, leaving invariant the line, can bring onto each other any two of its points (so that the line is "seen" from all its points in the same way). This property, which APOLLONIUS OF PERGA had called *homoeomerism*[35] does not characterize, however, the straight line, since it is shared by cylindrical helices and, among plane curves, by circumferences too. It could not have escaped EUCLID's notice that, in whatever way one tries to specify the meaning of the above sentence, circumferences (which surely *lie uniformly in respect to all their points*) seem difficult to exclude from this *definition*.

The definition of a straight line seems even more mysterious than that of trapezium which we have examined in Sect. 6. As in that case the mystery can however be unveiled with the help of HERON. His "definition" (HERON OF ALEXANDRIA, vol. IV, 16–18) begins as follows:

εὐθεῖα μὲν οὖν γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐπ′ αὐτῆς σημείοις κεῖται ὀρθὴ οὖσα καὶ οἷον ἐπ′ ἄκρον τεταμένη ἐπὶ τὰ πέρατα· ἥτις δύο δοθέντων σημείων μεταξὺ ἐλαχίστη ἐστὶν τῶν τὰ αὐτὰ πέρατα ἐχουσῶν γραμμῶν, ...

a straight line is [a line] *which, uniformly in respect to* [all] *its points, lies upright and stretched to the utmost towards the ends, such that, given two points, it is the shortest of the lines having them as ends, ...*

The origin of this characterization of straight line can be traced back, in all likelihood, to ARCHIMEDES (toward whom HERON always shows a great interest). ARCHIMEDES (*On the Sphere and Cylinder*, 10) had in fact assumed that among all lines with the same ends the straight line has the minimum length. It is worth noting that ARCHIMEDES' statement was not a "definition", but the first of the *postulates* (λαμβανόμενα) of the treatise. In order to draw a "definition" from ARCHIMEDES' postulate, HERON, however, could not restrict his statement to only one couple of points; he had to require that ARCHIMEDES' property should be verified *uniformly in respect to all its points*, i.e. ἐξ ἴσου τοῖς ἐπ' αὐτῆς σημείοις. HERON's sentence is therefore completely clear.

We know that the obscure scholar who compiled the list of definitions in the form in which they now appear in Book I of the *Elements* was not a mathematician of any value[36]. We have supposed that he had decided to use as *definitions* of elementary geometrical entities some excerpts from HERON's long illustrations. In our case he might have truncated HERON's first sentence as soon as he could get a syntactically correct sentence, even if empty of mathematical meaning. The circumstance that if we proceed in this way we get just the *definition* traditionally included in EUCLID's text (of which, on the other hand, no mathematician has ever been able to make any sense) gives a strong support to the above conjecture[37].

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[35] In the lost work Περὶ τοῦ κοχλίου (which is mentioned by PROCLUS, 105, 1–6) APOLLONIUS had proved that the cylindrical helix is just a *homoeomeric* curve.

[36] Cp. the remarks above, in Sect. 6, on definitions 15 and 22.

[37] The link between the "definition" appearing in the *Elements* and the rest of Heron's sentence is more apparent in the Greek text, since the verb *κεῖται* is constructed with participles only appearing in Heron. The above translation of Heron's passage is not very good, but a good English

Both PROCLUS (109–110) and SIMPLICIUS (see AN-NAIRĪZĪ,10) seem to retain, through channels very hard to reconstruct, some indirect memory of the connection between Archimedean postulates and the definitions of a straight line and a plane as included in the *Elements*. Both authors try in fact to persuade us, by offering strange arguments not so far understood, that the definitions included in the *Elements* essentially state the same minimum properties postulated by ARCHIMEDES.

It is worth remarking that SEXTUS EMPIRICUS, who quotes the definition of a straight line in the same truncated form in which it is now included in the *Elements*[38] and interprets it as the statement that the straight line lies *evenly* with respect to all its parts, does not fail to point out its obviously tautological nature[39].

Considerations completely analogous to the previous ones can be repeated about definition 7 (of plane). ARCHIMEDES' postulate concerning plane, the *definition* of a plane given by HERON and the truncated definition which appears in the *Elements* can all be obtained from the corresponding sentences concerning a straight line just replacing the words *straight line* and *point* with, respectively, *plane* and *straight line*.

11. Other testimonies

We have now to examine all ancient authors who quote some of the first seven definitions of the *Elements*. STAMATIS lists, as such *testimonia*, HERON, SEXTUS EMPIRICUS, PROCLUS, PHILOPONUS, PSELLUS, MARTIANUS CAPELLA and BOETHIUS (see STAMATIS, vol. I, xii-xxx, 1). We have discussed sufficiently the passages of HERON and SEXTUS EMPIRICUS. All other witnesses were active centuries after the corruption of EUCLID's text had occurred. Even though some of the above authors occasionally refer to better recensions of EUCLID's text than our manuscripts, we certainly cannot rely on their recognitions as a proof of authenticity[40]. If our interpretation of HERON's and SEXTUS' passages is accepted, there is no apparent attribution to EUCLID of the first seven defi-

translation could not contain as a subset the definition appearing in the *Elements*. A. Jones has suggested to me that the expression κεῖται ὀρθὴ οὖσα ... should probably be translated "is hypothesized to be erect...".

[38] See above, note 32. It is possible that SEXTUS, besides a genuine version of the *Elements*, also had at his disposal a list of truncated definitions by HERON, like the one reported in the Fayûm papyrus. Alternatively, it is conceivable that SEXTUS was induced to truncate HERON's definition by his interpretation.

[39] HEATH's translation of the sentence (*A straight line is a line which lies evenly with the points on itself*) was based on the same interpretation. The possibility of interpretating the Greek words ἐξ ἴσου as meaning *evenly* (instead of *uniformly*, as is clear in the complete HERON's text) could well have favoured the truncation of HERON's sentence, but only by generating a statement which is linguistically clear but mathematically meaningless. HEATH (vol. I, 167) concludes his analysis of the definition so interpretated saying that *the language is thus seen to be hopelessly obscure.*

[40] For PROCLUS see above, Sect. 5, in particular note 15. BOETHIUS, although occasionally, as in the case of the definition of circle, he reports better readings of EUCLID's work than the one of the manuscript tradition (see above, Sect. 2) is certainly not more reliable than PROCLUS. Probably none of these authors was provided with a better edition of the *Elements* than ours, but they could still read some old commentaries based on earlier recensions.

nitions of the Elements before late antiquity. This circumstance is consistent with our reconstruction, but, admittedly, like all arguments *ex silentio*, is a very weak support for it. We may well explain the silence of ancient testimonies on EUCLID about the subject of our definitions with the scant number of such testimonies (and also, one is tempted to say, with the uselessness of the definitions). This silence becomes, however, much more significant if it is considered jointly with another class of testimonies: the ancient writers (later than EUCLID and earlier than the extant recensions of the *Elements*) who do discuss definitions of fundamental mathematical entities. The point is that none of them mentions EUCLID[41], in spite of the general use of the *Elements* as the standard reference on elementary mathematics. In this second category of testimonies, besides SEXTUS EMPIRICUS (see above, Sects. 3 and 8) and IAMBLICHUS (see above, Sect. 7), I mention PLUTARCH, who has at least two opportunities to "define" the concept of straight line and in both occasions ignores the definition appearing in the *Elements*, but defines it as the shortest line connecting two points (*Platonicae Quaestiones*, 1003E; *De Pythiae oraculis*, 408F). Furthermore, when PLUTARCH has the opportunity to discuss the idea of point, he defines it as a *monad having position* (*Platonicae Quaestiones*, *1003F*), a very ancient definition (cp. above, Sect. 5), which, like the one of straight line just quoted, appears, among other sources, in HERON's work, but not in the *Elements*.

12. Conclusion

From the considerations so far exposed the following reconstruction emerges as the most likely possibility:

EUCLID had not inserted in his treatise the first seven definitions, leaving fundamental geometric entities undefined.

In the Imperial age, because of the decay of scientific methodology, EUCLID's choice could not be understood and the absence of the definitions of some elementary geometric entities seemed to be a lacuna of the *Elements*.

As a remedy for such a supposed lacuna, first HERON wrote his schoolbook (freely using any kind of sources) and later a list of excerpts of HERON 's work was compiled and (possibly later) interpolated into EUCLID's text.

The case considered in the present paper is just an example of a much more general phenomenon. We can know Hellenistic science only through the filter of later editors. They have not only often preserved only the most elementary works, but also have altered their text, adapting them to their own views, grounded on pre-Hellenistic philosophy (mainly on Platonic and Aristotelean views). As a result, Hellenistic science has been in many cases (as the one here considered[42]) so distorted that its being significantly underrated was almost unavoidable.

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[41] I have checked all the passages referring to (any) EUCLID contained in the TLG corpus, besides authors not included in the TLG, but known to me as authors interested in mathematical definitions.

[42] The thesis that an analogous phenomenon occurred in astronomy has been set out in RUSSO (1994).

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