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An Approximate Zero-One Law

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Summary. We prove an approximate zero-one law, which holds for finite Bernoulli schemes. An application to percolation theory is given.

1. Introduction

The present paper is an attempt to generalize some of the ideas recently developed in the particular context of the two-dimensional percolation theory.

We denote by μ_x the Bernoulli probability measure which assigns to each site of an infinite graph G the probability x to be "occupied". In percolation theory (introduced in [1]) one is interested to the μ_x-probability of some events, whose prototype is the event "there exists an infinite cluster of occupied sites". Since these events belong to the σ-algebra at infinity, $\mathscr{B}_\infty$, the usual zero-one law [2] can be applied, but it is not of help in characterizing the set of values of x for which the measure μ_x is supported by a given event in $\mathscr{B}_\infty$. Nevertheless, considerable progress has been recently made on this problem, in particular by proving, at least in some cases, the relation $p_c+p_c^*=1$ between the critical percolation probabilities of two dual graphs [3], [4], [5], [8]. The last relation is essentially equivalent to the statement that the event "simultaneous absence of infinite clusters of two opposite types" (corresponding, in a sense, to "critical behavior") has μ_x-probability zero, except for one and only one value of x ("critical percolation point").

The interest of this result is increased, in our opinion, by the circumstance that it has been obtained without solving the model (actually the value of the critical percolation point is still unknown for many graphs).

An essential tool in getting the above results has been the introduction of a finite analogous problem ("sponge percolation theory", see [6], [7]); the proofs, however, are heavily based on geometrical techniques, typical of the particular nature of the problem.

The aim of this paper is to show that some of these techniques are just a particular case of a general theorem, namely an "approximate zero-one law" which holds for local events which, in the sense we shall specify below, approximate the events in the σ-algebra at infinity.

The main interest of this theorem consists, perhaps, in the fact that, whereas the usual zero-one law holds for each measure μ_x individually considered,

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the approximate version of it is a statement about the whole family $\{\mu_x\}_{x\in[0,1]}$. This feature of the "approximate zero-one law", in particular, clarifies the role of the finite graphs in percolation theory; the last section is devoted to a new proof of the relation $p_c+p_c^*=1$, obtained as a corollary of our main theorem.

Our hope, of course, is that Theorem 1 can be useful in getting new results, in particular in the ν-dimensional percolation theory; furthermore we think that possible generalizations of Theorem 1 to finite volume Gibbs measures could be useful in the theory of phase transitions.

2. Definitions and Statement of the Results

We consider the space $\Omega=\{-1,1\}^L$, where L is a countable set. For every $i\in L$ we put $E_i^+[E_i^-]=\{\omega\in\Omega|\omega(i)=1[-1]\}$. We call $\mathscr{B}$ the σ-algebra generated by the events E_i^+, $i\in L$, and, for any $K\subset L$, we call $\mathscr{B}_K$ the σ-algebra generated by the events E_i^+, $i\in K$. We define in Ω the partial order $\leqq$ by putting $\omega_1\leqq\omega_2$ if and only if $\forall\, i\in L\ \ \omega_1(i)\leqq\omega_2(i)$ and we call positive an event $A\in\mathscr{B}$ if its characteristic function is non-decreasing.

For any $x\in[0,1]$ we consider the Bernoulli probability measure

$$\mu_x=\prod_{i\in L}\nu_x,$$

where ν_x is the measure on $\{-1,1\}$ which assigns weights x and $1-x$ respectively to 1 and -1. The σ-algebra at infinity, $\mathscr{B}_\infty$, is defined by

$$\mathscr{B}_\infty=\bigcap_{K\in\mathscr{F}}\mathscr{B}_{L\setminus K},$$

where $\mathscr{F}$ is the family of the finite subsets of L. Furthermore we consider the family of events

$$\mathscr{B}_F=\bigcup_{K\in\mathscr{F}}\mathscr{B}_K.$$

For any $i\in L$ we define $S_i\colon\Omega\to\Omega$ by putting

$$(S_i\omega)(i)=-\omega(i);\qquad \forall\, k\neq i\ \ (S_i\omega)(k)=\omega(k).$$

If $i\in L$, $A\in\mathscr{B}$ we put

$$\delta_i^I A=\{\omega\in A|S_i\omega\notin A\};\qquad \delta_i^E A=\{\omega\in\Omega\setminus A|S_i\omega\in A\};\qquad \delta_i A=\delta_i^I A\cup\delta_i^E A.$$

If $\omega\in\delta_i A$ we call i a pivotal site for the configuration ω and for the event A; the set

$$C_A(\omega)=\{i\in L|\omega\in\delta_i A\}$$

is called the pivotal set of the configuration ω for the event A; furthermore we shall consider the events

$$\delta^I A=\bigcup_{i\in L}\delta_i^I A;\qquad \delta^E A=\bigcup_{i\in L}\delta_i^E A;\qquad \delta A=\delta^I A\cup\delta^E A.$$

The well known zero-one law, in our setting, states that if $A \in \mathscr{B}_\infty$, then for any $x \in [0,1]$ $\mu_x(A)$ equals either zero or one. Our purpose is to prove an approximate form of this statement for a suitable class of events in $\mathscr{B}_F$. In order to characterize this class, we observe that the following proposition holds:

Proposition 1. *An event $A \in \mathscr{B}$ belongs to $\mathscr{B}_\infty$ if and only if for every $i \in L$ $\delta_i A = \emptyset$.*

Proposition 1, whose proof consists in a direct application of the definitions, suggests that the number $\operatorname*{Sup}_{x,i} \mu_x(\delta_i A)$ could be a measure of the "distance" of the event A from $\mathscr{B}_\infty$. Our main result is the following theorem, which shows that, under the additional hypothesis of positivity, the events in $\mathscr{B}_F$ which are "near" to $\mathscr{B}_\infty$ (in the above sense) satisfy an approximate zero-one law.

Theorem 1. *For every $\varepsilon > 0$, there exists $\eta > 0$ such that if $A \in \mathscr{B}_F$ is a positive event and*

$$\forall i \in L, \ \forall x \in [0,1], \qquad \mu_x(\delta_i A) < \eta, \tag{2.1}$$

then there exists $x_0 \in [0,1]$ such that

$$\forall x \leqq x_0 - \varepsilon \qquad \mu_x(A) \leqq \varepsilon, \tag{2.2}$$

$$\forall x \geqq x_0 + \varepsilon \qquad \mu_x(A) \geqq 1 - \varepsilon. \tag{2.3}$$

If we replace (2.1) by the stronger condition

$$\forall x \in [0,1] \qquad \mu_x(\delta A) < \eta, \tag{2.1'}$$

we obtain a weaker theorem whose proof is much simpler than the one of Theorem 1. We start by proving this last statement in the next section.

3. Proof of a Weaker Statement

If ν, μ are two probability measures defined on $\mathscr{B}$ we write $\nu \overset{\varepsilon}{\leqq} \mu$ if there exists a probability measure m on $\Omega \times \Omega$ such that

a) m is a joint representation of ν and μ, i.e. $\forall B \in \mathscr{B}$,

$$m\{(\omega, w) \in \Omega \times \Omega \mid \omega \in B\} = \nu(B), \qquad m\{(\omega, w) \in \Omega \times \Omega \mid w \in B\} = \mu(B).$$

b) $m\{(\omega, w) \in \Omega \times \Omega \mid \omega \leqq w\} \geqq 1 - \varepsilon$.

Remark. The above definition implies that if $\nu \overset{\varepsilon}{\leqq} \mu$ and A is a positive event, then $\nu(A) \leqq \mu(A) + \varepsilon$.

In this section we identify L with the set N of the positive integers (since L is a countable set there is no loss of generality in this assumption). If $w \in \Omega$, $k \in N$, we denote by $w^{(k)}$ the cylinder

$$w^{(k)} = \{\omega \in \Omega \mid \forall i \leqq k \ \omega(i) = w(i)\};$$

furthermore we put

$$w^{(0)} = \Omega.$$

Lemma 1. *If ν is a probability measure on $\mathscr{B}$, and $C \in \mathscr{B}$ is an event such that*

α) $\nu(C) \geqq 1-\varepsilon$,

β) $\forall w \in C$, $\forall k \in N$ $\nu(E_k^+ | w^{(k-1)}) \leqq x$,

then $\nu \overset{\varepsilon}{\leqq} \mu_x$.

Proof. We define recursively a measure m on $\Omega \times \Omega$ by putting:

A) $m(E_1^+ \times E_1^+) = \nu(E_1^+)$; $m(E_1^+ \times E_1^-) = 0$;
$m(E_1^- \times E_1^+) = x - \nu(E_1^+)$; $m(E_1^- \times E_1^-) = 1 - x$;

B) if $k \geqq 2$ and $w^{(k-1)}$ satisfies the inequality $\nu(E_k^+ | w^{(k-1)}) \leqq x$, then, for any $\omega \in \Omega$,

$$m(E_k^+ \times E_k^+ | w^{(k-1)} \times \omega^{(k-1)}) = \nu(E_k^+ | w^{(k-1)});$$
$$m(E_k^+ \times E_k^- | w^{(k-1)} \times \omega^{(k-1)}) = 0;$$
$$m(E_k^- \times E_k^+ | w^{(k-1)} \times \omega^{(k-1)}) = x - \nu(E_k^+ | w^{(k-1)});$$
$$m(E_k^- \times E_k^- | w^{(k-1)} \times \omega^{(k-1)}) = 1 - x;$$

C) if $k \geqq 2$ and $\nu(E_k^+ | w^{(k-1)}) > x$, then for any $\sigma, \sigma' \in \{-1, 1\}$, and for any $\omega \in \Omega$,

$$m(E_k^\sigma \times E_k^{\sigma'} | w^{(k-1)} \times \omega^{(k-1)}) = \nu(E_k^\sigma | w^{(k-1)}) \, \mu_x(E_k^{\sigma'}).$$

It is easy to verify that A), B), C) define a probability measure m on $\Omega \times \Omega$ which is a joint representation of ν and μ_x. Furthermore

$$m\{(\omega, w) \in \Omega \times \Omega | \omega \leqq w\} \geqq \nu(C) \geqq 1 - \varepsilon.$$

Hence $\nu \overset{\varepsilon}{\leqq} \mu_x$.

Proposition 2. *If $A \in \mathscr{B}$ is a positive event, $\mu_x(A) \geqq \varepsilon$, $\mu_x(\delta^I A) < \varepsilon^3$, then $\mu_{x+\varepsilon}(A) \geqq 1 - \varepsilon$.*

Proof. We put $\mu_{x,A}(\cdot) = \mu_x(\cdot | A)$. It suffices to prove that

$$\mu_{x,A} \overset{\varepsilon}{\leqq} \mu_{x+\varepsilon}, \tag{3.1}$$

(3.1) and the Remark, indeed, imply that $1 = \mu_{x,A}(A) \leqq \mu_{x+\varepsilon}(A) + \varepsilon$. We consider the events

$$C_k = \{w \in \Omega | \mu_x(\delta_k^I A | w^{(k-1)}) \leqq \varepsilon \mu_x(A | w^{(k-1)})\}; \qquad C = \bigcap_{k=1}^{\infty} C_k.$$

For the proof of (3.1) it is enough, in view of Lemma 1, to prove the following two inequalities:

$$\mu_{x,A}(C) \geqq 1 - \varepsilon; \tag{3.2}$$
$$\forall w \in C, \ \forall k \in N, \quad \mu_{x,A}(E_k^+ | w^{(k-1)}) \leqq x + \varepsilon. \tag{3.3}$$

Since $\mu_x(A) \geqq \varepsilon$, we have:

$$\mu_{x,A}(\Omega \setminus C) \leqq \frac{1}{\varepsilon} \mu_x(A \cap (\Omega \setminus C)) = \frac{1}{\varepsilon} \sum_{k=1}^{\infty} \mu_x(A \cap D_k),$$

where

$$D_k = \left(\bigcap_{i=1}^{k-1} C_i \right) \cap (\Omega \setminus C_k).$$

The definitions of C_k, D_k imply that $\mu_x(\delta_k^I A|D_k)>\varepsilon\mu_x(A|D_k)$. By using the last inequality and the remark that the events D_k are pairwise disjoint we get

$$\mu_{x,A}(\Omega\setminus C)<\varepsilon^{-2}\sum_{k=1}^{\infty}\mu_x(\delta_k^I A\cap D_k)\leqq\varepsilon^{-2}\sum_{k=1}^{\infty}\mu_x(\delta^I A\cap D_k)\leqq\varepsilon^{-2}\mu_x(\delta^I A).$$

Since $\mu_x(\delta^I A)<\varepsilon^3$, we get (3.2).

Now we observe that, for any $w\in\Omega$, $k\in N$,

$$\mu_{x,A}(E_k^+|w^{(k-1)})=[\mu_x(A\cap w^{(k-1)})]^{-1}\cdot[\mu_x(E_k^+\cap(A\setminus\delta_k^I A)\cap w^{(k-1)})+\mu_x(E_k^+\cap\delta_k^I A)\cap w^{(k-1)}].$$

Since A is a positive event, $\delta_k^I A\subset E_k^+$; furthermore the events E_k^+ and $(A\setminus\delta_k^I A)\cap w^{(k-1)}$ are μ_x-independent. Hence

$$\begin{aligned}\mu_{x,A}(E_k^+|w^{(k-1)})&=x\frac{\mu_x((A\setminus\delta_k^I A)\cap w^{(k-1)})}{\mu_x(A\cap w^{(k-1)})}+\frac{\mu_x(\delta_k^I A\cap w^{(k-1)})}{\mu_x(A\cap w^{(k-1)})}\\&\leqq x+\frac{\mu_x(\delta_k^I A\cap w^{(k-1)})}{\mu_x(A\cap w^{(k-1)})}.\end{aligned}$$

The last inequality and the definition of C imply (3.3). This ends the proof of Proposition 2.

Proposition 2, in particular, implies that if a positive event $A\in\mathscr{B}_F$ satisfies the condition

$$\forall x\in[0,1]\qquad \mu_x(\delta A)<\varepsilon^3, \tag{2.1''}$$

then the statement of Theorem 1 holds. Roughly speaking, the proof of Theorem 1 shall be completed by proving that if for each i the μ_x-probability of the event $\delta_i A$ is very small, then the event $A=\bigcup_i \delta_i A$, too, has a small μ_x-probability, except, at most, for a "small" set of values of x. In order to prove this (at first sight surprising) statement we need to improve our knowledge of the properties of the pivotal set. This shall be done in the next section.

4. Some Lemmas about the Expected Size of the Pivotal Set

The number of pivotal sites for the configuration ω for the event A is, of course:

$$n_A(\omega)=\sum_{i\in L}\chi_{\delta_i A}(\omega), \tag{4.1}$$

where we have used the symbol χ_E for the characteristic function of the event E. Furthermore we put

$$n_A^I(\omega)=\sum_{i\in L}\chi_{\delta_i^I A}(\omega);\qquad n_A^E(\omega)=\sum_{i\in L}\chi_{\delta_i^E A}(\omega).$$

In this section we collect some lemmas about the expectation of the random variable n_A. We shall denote by E_x the expectation with respect to the measure μ_x. We start by recalling a simple equality which was used in [5].

Lemma 2. *If $A \in \mathscr{B}_F$ is a positive event* $E_x n_A = \frac{d}{dx} \mu_x(A)$.

Proof. See Lemma 3 of [5].

Lemma 3. *For every $A \in \mathscr{B}_F$, $A \neq \emptyset$, $E_x(n_A | A) \geqq \log_{x'} \mu_x(A)$, where* $x' = \min(x, 1-x)$.

The proof of Lemma 3 is based on the elementary inequality stated in the following lemma.

Lemma 4. *If $x, \alpha, \beta \in (0,1)$, then*

$$\log_{x'}[x\alpha + (1-x)\beta] \leqq \frac{x\alpha}{x\alpha+(1-x)\beta} \log_{x'} \alpha + \frac{(1-x)\beta}{x\alpha+(1-x)\beta} \log_{x'} \beta + \frac{x'|\alpha-\beta|}{x\alpha+(1-x)\beta}.$$

Proof. We can suppose, without loss of generality, $\beta \leqq \alpha$; then it suffices to prove that for any $\alpha \in (0,1)$ the function

$$\begin{aligned} f_\alpha(\beta) = {} & [x\alpha + (1-x)\beta]^{-1} [x\alpha \log_{x'} \alpha + (1-x)\beta \log_{x'} \beta + x'(\alpha - \beta)] \\ & - \log_{x'}[x\alpha + (1-x)\beta] \end{aligned}$$

is non-negative for $\beta \in [0, \alpha]$. Elementary computations show that

$$\frac{d}{d\beta} f_\alpha(\beta) = x'\alpha[x\alpha + (1-x)\beta]^{-2}[(1-x')\log_{x'}(\beta/\alpha) - 1].$$

The function $\frac{d}{d\beta} f_\alpha(\beta)$ equals zero only in the point $\beta' = \alpha x'^{1/(1-x')} < \alpha$; it is positive for $\beta \in (0, \beta')$ and it is negative for $\beta \in (\beta', \alpha)$. Since $f_\alpha(0) = x'/x - \log_{x'} x \geqq 0$, $f_\alpha(\alpha) = 0$, we get $f_\alpha(\beta) \geqq 0$ for $\beta \in [0, \alpha]$. This proves Lemma 4.

Proof of Lemma 3. First we suppose that A is a cylinder, i.e.

$$A = \{\omega \in \Omega \,|\, \forall\, i \in \Lambda\, \omega(i) = \sigma(i)\},$$

where Λ is a finite subset of L and $\sigma \in \{-1, 1\}^\Lambda$. If we denote by $|\Lambda|$ the number of elements of Λ, we have

$$E_x(n_A | A) = |\Lambda|; \quad \mu_x(E) = \prod_{i \in \sigma^{-1}(1)} x \prod_{i \in \sigma^{-1}(-1)} (1-x) \geqq x'^{|\Lambda|};$$

hence in this case Lemma 3 holds. Now we prove Lemma 3, by induction on $|\Lambda|$, for an arbitrary event $A \in \mathscr{B}_\Lambda$. If $|\Lambda| = 0$, since $A \neq \emptyset$, we have $A = \Omega$ and both sides of the inequality stated by Lemma 3 are zero. Suppose $|\Lambda| > 0$ and let i be an element of Λ. Then A can be written, in an unique way, as

$$A = (E_i^+ \cap A_i^+) \cup (E_i^- \cap A_i^-),$$

where A_i^+, $A_i^- \in \mathscr{B}_{\Lambda \setminus \{i\}}$. If for every $i \in \Lambda$ one of the two events A_i^+, A_i^- is empty, then it is clear that A is a cylinder. Hence we can suppose $A_i^+ \neq \emptyset$, $A_i^- \neq \emptyset$. Then we have $\forall j \neq i$

$$\delta_j^I A = [E_i^+ \cap \delta_j^I A_i^+] \cup [E_i^- \cap \delta_j^I A_i^-];$$

$$\delta_i^I A = [E_i^+ \cap (A_i^+ \setminus A_i^-)] \cup [E_i^- \cap (A_i^- \setminus A_i^+)];$$

$$\begin{aligned}E_x(n_A|A)&=E_x(n_A^I|A)=[\mu_x(A)]^{-1}\sum_{j\in A}\mu_x(\delta_j^I A)\\&=[\mu_x(A)]^{-1}\Big[\sum_{j\in A\setminus\{i\}}(x\mu_x(\delta_j^I A_i^+)+(1-x)\,\mu_x(\delta_j^I A_i^-))\\&\quad+x\mu_x(A_i^+\setminus A_i^-)+(1-x)\,\mu_x(A_i^-\setminus A_i^+)\Big]\\&=[\mu_x(A)]^{-1}[x\mu_x(A_i^+)\,E_x(n_{A_i^+}|A_i^+)+(1-x)\,\mu_x(A_i^-)\,E_x(n_{A_i^-}|A_i^-)\\&\quad+x\mu_x(A_i^+\setminus A_i^-)+(1-x)\,\mu_x(A_i^-\setminus A_i^+)].\end{aligned}$$

Since

$$\mu_x(A)=x\mu_x(A_i^+)+(1-x)\,\mu_x(A_i^-),$$

$$x\mu_x(A_i^+\setminus A_i^-)+(1-x)\,\mu_x(A_i^-\setminus A_i^+)\geqq x'\mu_x(A_i^+\,\Delta A_i^-)\geqq x'|\mu_x(A_i^+)-\mu_x(A_i^-)|,$$

we get

$$E_x(n_A|A)\geqq\frac{x\mu_x(A_i^+)\,E_x(n_{A_i^+}|A_i^+)+(1-x)\,\mu_x(A_i^-)\,E_x(n_{A_i^-}|A_i^-)+x'|\mu_x(A_i^+)-\mu_x(A_i^-)|}{x\mu_x(A_i^+)+(1-x)\,\mu_x(A_i^-)}.$$

By recurrence, we get

$$E_x(n_A|A)\geqq\frac{x\mu_x(A_i^+)\log_{x'}\mu_x(A_i^+)+(1-x)\,\mu_x(A_i^-)\log_{x'}\mu_x(A_i^-)+x'|\mu_x(A_i^+)-\mu_x(A_i^-)|}{x\mu_x(A_i^+)+(1-x)\,\mu_x(A_i^-)}.$$

By using Lemma 4 finally we get

$$E_x(n_A|A)\geqq\log_{x'}[x\mu_x(A_i^+)+(1-x)\,\mu_x(A_i^-)]=\log_{x'}\mu_x(A).$$

Lemma 3, roughly speaking, implies that if an event A has a small μ_x-probability and x is bounded away from zero and one, then $E_x(n_A|A)$ must be large.

The converse statement is in general false, but it holds (in the weak sense specified by the following lemma) for the class of events which we shall consider below.

We call $\mathscr{S}$ the family of events $S\in\mathscr{B}_F$ of the type $S=A\setminus B$, where $A, B\in\mathscr{B}_F$ are positive events.

Remark. An event $S\in\mathscr{B}_F$ belongs to $\mathscr{S}$ if and only if it satisfies the following condition

(α) if $\omega_1\leqq\omega_2\leqq\omega_3$, $\omega_1\in S$, $\omega_2\notin S$, then $\omega_3\notin S$.

Proof of the remark. For any event S we put $\hat{S}=\{\omega\in\Omega\setminus S|\exists w: w\in S\}$. The event $S\cup\hat{S}$ is obviously a positive event and $S=(S\cup\hat{S})\setminus\hat{S}$. Since condition (α) means that $\hat{S}$ is a positive event, if (α) holds we get $S\in\mathscr{S}$. The proof of the converse is left to the reader.

Lemma 5. If $0<\alpha<\beta<1$, $S\in\mathscr{S}$, then

$$\int_\alpha^\beta\mu_x(S)\,dx\leqq 2\big[\operatorname*{Inf}_{x\in[\alpha,\beta]}E_x(n_S|S)\big]^{-1}.$$

Proof. Suppose $S=A\setminus B$, where $A, B\in\mathscr{B}_F$ are positive events. For any $i\in L$ we have $\delta_i^I A\subset E_i^+$, $\delta_i^E B\subset E_i^-$; hence

$$\delta_i^I A \cap \delta_i^E B = \emptyset. \tag{4.1}$$

Furthermore:

$$\delta_i^I S = (\delta_i^I A \cap S) \cup (\delta_i^E B \cap S). \tag{4.2}$$

(4.1) and (4.2) imply that, for each $\omega \in S$,

$$n_S(\omega) = n_S^I(\omega) = n_A^I(\omega) + n_B^E(\omega) = n_A(\omega) + n_B(\omega).$$

Hence we have

$$\mu_x(S) E_x(n_S | S) = \int_S (n_A + n_B) d\mu_x \leqq E_x(n_A + n_B)$$

$$\mu_x(S) \operatorname*{Inf}_{x \in [\alpha, \beta]} E_x(n_S | S) \leqq E_x n_A + E_x n_B.$$

Since A, B are positive events, Lemma 2 implies that

$$\operatorname*{Inf}_{x \in [\alpha, \beta]} E_x(n_S | S) \int_\alpha^\beta \mu_x(S)\, dx \leqq [\mu_\beta(A) - \mu_\alpha(A)] + [\mu_\beta(B) - \mu_\alpha(B)] \leqq 2.$$

Our use of Lemma 5 is based on the remark contained in the following lemma.

Lemma 6. *If $A \in \mathscr{B}_F$ is a positive event, for every integer $k > 0$, the events*

$$\Delta_k^I A = \{\omega \in A | n_A^I(\omega) = k\}, \qquad \Delta_k^E A = \{\omega \in \Omega \setminus A | n_A^E(\omega) = k\}$$

belong to $\mathscr{S}$.

Proof. It is easy to verify that, since A is positive, $n_A^I [n_A^E]$, restricted to $A [\Omega \setminus A]$ is a non-increasing [non-decreasing] function. Hence, if

$$\omega_1 \leqq \omega_2 \leqq \omega_3, \qquad \omega_1 \in \Delta_k^I A [\omega_1 \in \Delta_k^E A], \qquad \omega_2 \notin \Delta_k^I A [\omega_2 \notin \Delta_k^E A],$$

then $n_A^I(\omega_2) < k$ [either $n_A^E(\omega_2) > k$ or $\omega_2 \in A$]; therefore $\omega_3 \notin \Delta_k^I A [\omega_3 \notin \Delta_k^E A]$. Hence $\Delta_k^I A$ and $\Delta_k^E A$ satisfy the condition (α) of the above remark.

5. Proof of the Theorem

In this section we prove Theorem 1. Its proof is based on Proposition 2 and on the following lemma.

Lemma 7. *If $A \in \mathscr{B}_F$ is a positive event, then, for any $\alpha \in (0, 1/2)$,*

$$\int_\alpha^{1-\alpha} \mu_x(\delta A)\, dx \leqq 4 [\log_\alpha \eta]^{-1/2}, \qquad \text{where } \eta = \operatorname*{Max}_{i \in L} \operatorname*{Max}_{x \in [0,1]} \mu_x(\delta_i A).$$

Proof. Lemmas 5 and 6 imply that

$$\int_\alpha^{1-\alpha} \mu_x(\Delta_k^I A)\, dx \leqq 2 [\operatorname*{Inf}_{x \in (\alpha, 1-\alpha)} E_x(n_{\Delta_k^I A} | \Delta_k^I A)]^{-1}. \tag{5.1}$$

On the other hand we have

$$E_x(n_{\Delta_k^I A} | \Delta_k^I A) = [\sum_{i_1 \ldots i_k} \mu_x(\Delta_{i_1 \ldots i_k}^I A)]^{-1} \sum_{i_1 \ldots i_k} E_x(n_{\Delta_k^I A} | \Delta_{i_1 \ldots i_k}^I A) \mu_x(\Delta_{i_1 \ldots i_k}^I A) \tag{5.2}$$

where

$$\Delta^I_{i_1\dots i_k}A=\{\omega\in\delta^I A\,|\,C_A(\omega)=\{i_1\dots i_k\}\}.$$

Now we claim that

$$E_x(n_{\Delta^I_k A}\,|\,\Delta^I_{i_1\dots i_k}A)=E_x(n_{\Delta^I_{i_1\dots i_k}A}\,|\,\Delta^I_{i_1\dots i_k}A). \tag{5.3}$$

We remark that $C_A(\omega)$, restricted to $\delta^I A$, is a non-increasing function (if the pivotal sets are ordered by inclusion); this remark implies that, if $\omega\in\Delta^I_{i_1\dots i_k}A$, $j\in L$, then $S_j\omega$ belongs to $\Delta^I_k A$ if and only if it belongs to $\Delta^I_{i_1\dots i_k}A$. This proves the claim. Lemma 3, applied to the r.h.s. of (5.3) yields

$$\forall\, i_1,\dots,i_k\in L\qquad E_x(n_{\Delta^I_k A}\,|\,\Delta^I_{i_1\dots i_k}A)\geqq\log_{x'}\eta, \tag{5.4}$$

where we have used the obvious inclusion $\Delta^I_{i_1\dots i_k}A\subset\delta_{i_1}A$. By collecting together (5.1), (5.2), (5.4) we get

$$\int_\alpha^{1-\alpha}\mu_x(\Delta^I_k A)\,dx\leqq 2[\log_\alpha\eta]^{-1}. \tag{5.5}$$

In an analogous way one can prove the inequality:

$$\int_\alpha^{1-\alpha}\mu_x(\Delta^E_k A)\,dx\leqq 2[\log_\alpha\eta]^{-1}. \tag{5.5'}$$

By summing (5.5) and (5.5′) we get

$$\forall\, k\in N\qquad \int_\alpha^{1-\alpha}\mu_x(\Delta_k A)\,dx\leqq 4[\log_\alpha\eta]^{-1}, \tag{5.6}$$

where

$$\Delta_k A=\Delta^E_k A\cup\Delta^I_k A.$$

On the other hand, since A is a positive event, from Lemma 2 we get

$$\sum_{k=1}^{\infty}k\int_\alpha^{1-\alpha}\mu_x(\Delta_k A)\,dx=\int_\alpha^{1-\alpha}E_x n_A\,dx\leqq 1. \tag{5.7}$$

We denote by k_0 the integral part of $\frac{1}{2}[\log_\alpha\eta]^{1/2}$; (5.6) and (5.7) yield

$$\begin{aligned}\int_\alpha^{1-\alpha}\mu_x(\delta A)\,dx&=\sum_{k=1}^{k_0}\int_\alpha^{1-\alpha}\mu_x(\Delta_k A)\,dx+\sum_{k=k_0+1}^{\infty}\int_\alpha^{1-\alpha}\mu_x(\Delta_k A)\,dx\\&\leqq\tfrac{1}{2}[\log_\alpha\eta]^{1/2}4[\log_\alpha\eta]^{-1}+\sum_{k=1}^{\infty}\frac{k}{k_0+1}\int_\alpha^{1-\alpha}\mu_x(\Delta_k A)\,dx\\&\leqq 2[\log_\alpha\eta]^{-1/2}+2[\log_\alpha\eta]^{-1/2}\sum_{k=1}^{\infty}k\int_\alpha^{1-\alpha}\mu_x(\Delta_k A)\,dx\\&\leqq 4[\log_\alpha\eta]^{-1/2}.\end{aligned}$$

Proof of Theorem 1. Let $0<\varepsilon<1/2$. We choose η by putting

$$4[\log_{\varepsilon/2}\eta]^{-1/2}=\varepsilon^4/2. \tag{5.8}$$

Let $A \in \mathscr{B}_F$ a positive event such that $\forall i \in L$, $\forall x \in [0,1]$, $\mu_x(\delta_i A) \leqq \eta$. We put

$$\hat{x} = \mathrm{Sup}\,\{x \in [0,1] \mid \mu_x(A) < \varepsilon\}; \qquad \bar{x} = \mathrm{Max}\,\{\hat{x}, \varepsilon/2\}.$$

We can suppose $\bar{x} < 1-\varepsilon$ (otherwise the statement of the theorem obviously holds with the choice $x_0 = \hat{x}$); then Lemma 7 yields:

$$\int_{\bar{x}}^{\bar{x}+\varepsilon/2} \mu_x(\delta A)\,dx \leqq \int_{\varepsilon/2}^{1-\varepsilon/2} \mu_x(\delta A)\,dx \leqq 4[\log_{\varepsilon/2}\eta]^{-1/2} = \varepsilon^4/2.$$

The last inequality implies that there exists $x_0 \in (\bar{x}, \bar{x}+\varepsilon/2)$ such that $\mu_{x_0}(\delta A) \leqq \varepsilon^3$; furthermore, since $x_0 > \bar{x} \geqq \hat{x}$, we have $\mu_{x_0}(A) \geqq \varepsilon$. Hence Proposition 2 implies that $\mu_{x_0+\varepsilon}(A) > 1-\varepsilon$. On the other hand, since $x_0 - \varepsilon < \bar{x} - \varepsilon/2 \leqq \hat{x}$, the definition of $\hat{x}$ implies $\mu_{x_0-\varepsilon}(A) < \varepsilon$.

6. An Application to Percolation Theory

The relation

$$p_c + p_c^* = 1 \tag{6.1}$$

between the critical percolation probabilities of a pair of dual planar graphs has been proved, recently, in a variety of instances [3], [5], [8]. In this section, as a first example of application of Theorem 1, we sketch a new proof of (6.1) in the case of the site percolation problem on the two-dimensional square lattice.

We use the same notations as in [5]; in particular $A_{L,1}^{+}$ is the event "there exists a $(+)$ chain in Λ_L connecting its left side with its right side", where $\Lambda_L = \{i \in Z^2 \mid |i_1| \leqq L,\ |i_2| \leqq L\}$.

Lemma 8. *$\forall L, \forall x \in (1-p_c^*, p_c)$, $\beta \leqq \mu_x(A_{L,1}^{+}) \leqq 1-\beta$, where β is the root in $[0,1]$ of the equation $x^3[1-(1-x)^{1/2}]^{12} = 1-5^{-4}$.*

Proof. See Lemma 5 of [5].

Theorem 2. $p_c + p_c^* = 1$.

Proof. It is easy to verify that

$$\delta_i A_{L,1}^{+} \subseteq (\Omega \setminus F_{L,i}^{+}) \cap (\Omega \setminus F_{L,i}^{-*}),$$

where $F_{L,i}^{+}[F_{L,i}^{-*}] = \{\omega \in \Omega \mid i$ is surrounded in ω by a $(+)[(-*)]$ circuit either internal to Λ_L or intersecting Λ_L in at most two adjacent sides$\}$. Furthermore, by using translation invariance of μ_x, it is easy to convince oneself that

$$\forall i \in \Lambda_L,\ \forall x \in [0,1], \qquad \mu_x(F_{L,i}^{+} \cup F_{L,i}^{-*}) \geqq \mu_x(E_L^{+} \cup E_L^{-*}),$$

where

$E_L^{+}[E_L^{-*}] = \{\omega \in \Omega \mid O$ is surrounded in ω by a $(+)[(-*)]$ circuit internal to $\Lambda_L\}$.

Hence we have

$$\forall x \in [0,1],\ \forall i \in Z^2, \qquad \mu_x(\delta_i A^+_{L,1}) \leqq 1 - \mu_x(E^+_L \cup E^{-*}_L). \tag{6.2}$$

Now we suppose $p_c + p^*_c > 1$ (it is known [9] that $p_c + p^*_c \geqq 1$). Then we can choose π such that $1 - p^*_c < \pi < p_c$. The definitions of p_c, p^*_c imply that μ_π – a.s. there are neither infinite $(+)$ cluster nor infinite $(-*)$ clusters; hence:

$$\lim_{L\to\infty} \mu_\pi(E^+_L) = \lim_{L\to\infty} \mu_\pi(E^{-*}_L) = 1. \tag{6.3}$$

Since $\mu_x(E^+_L)[\mu_x(E^{-*}_L)]$ is an increasing [decreasing] function of x, we have:

$$\forall x \in [0,1],\ \forall L > 0, \qquad \mu_x(E^+_L \cup E^{-*}_L) \geqq \min\{\mu_\pi(E^+_L), \mu_\pi(E^{-*}_L)\}. \tag{6.4}$$

(6.2), (6.3) and (6.4) yield

$$\lim_{L\to\infty} \operatorname*{Max}_{i\in A_L} \operatorname*{Max}_{x\in[0,1]} \mu_x(\delta_i A^+_{L,1}) = 0. \tag{6.5}$$

The last relation, together with Theorem 1, implies that if L is large enough there exists an interval $(x_1, x_2) \subset (0,1)$ such that:

i) $x_2 - x_1 < p_c + p^*_c - 1$;

ii) $\forall x < x_1$, $\mu_x(A^+_{L,1}) < \beta$;

iii) $\forall x > x_2$, $\mu_x(A^+_{L,1}) > 1 - \beta$;

where β is the number defined in Lemma 8. i), ii), iii) are in contradiction with Lemma 8. This proves Theorem 2.

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