Forum Math. 3 (1991), 401–414

Forum
Mathematicum


# Bernoulli and Gibbs Probabilities of Subgroups of $\{0, 1\}^S$

Camillo Cammarota and Lucio Russo

(Communicated by Giovanni Gallavotti)

**Abstract.** We consider Bernoulli measures of parameter $x \in [0, 1/2]$ on the space $\{0, 1\}^S$ where $S$ is a finite set. We prove some new correlation inequalities and monotonicity properties of these measures, related to the natural group structure of the space. One peculiar feature of these inequalities is that they are preserved by conditioning the Bernoulli measures to a subgroup; in this way we can show that some basic techniques in Statistical Mechanics naturally fit in this scheme.

1980 Mathematical Subjects Classification (1985 Revision): 82A05; 60B15.

## 1. Definitions and main results

We are interested in probability measures on the space $\Omega_S = \{0, 1\}^S$, where $S$ is a finite set. In particular we shall consider the Bernoulli probability measure of parameter $x$, which we shall denote by $\mu_x$.

The set $\Omega_S$ has a natural order structure, which allows to define the notion of increasing (or decreasing) events. Harris [1] first remarked that if the events $E_1$, $E_2$ are both increasing (or both decreasing) then

$$(1.1) \qquad \mu_x(E_1 \cap E_2) \geq \mu_x(E_1)\mu_x(E_2) .$$

Fortuin, Kasteleyn and Ginibre [2] generalized Harris' inequality (1.1). The notion of F.K.G. measure and F.K.G. order between measures can be summarized as follows. A probability measure $\mu$ is F.K.G. if for any two increasing events $E_1$ and $E_2$

$$(1.1\text{a}) \qquad \mu(E_1 \cap E_2) \geq \mu(E_1)\mu(E_2) .$$

It was proved in [2] that a sufficient condition for the F.K.G. property is the following inequality (which can be easily verified in the case of Bernoulli measures):

$$(1.2) \qquad \forall \sigma_1, \sigma_2 \in \Omega_S \quad \mu(\sigma_1 \cup \sigma_2)\mu(\sigma_1 \cap \sigma_2) \geq \mu(\sigma_1)\mu(\sigma_2) ,$$

(where we have identified the elements of $\Omega_S$ with the subsets of $S$). Besides Bernoulli measures, an important class of F.K.G. measures which satisfy (1.2) can be obtained by considering finite volume Gibbs measures; for example if $S$ is an hypercube in $Z^v$ condition (1.2) is satisfied by the Gibbs measures of the ferromagnetic n.n. Ising model at external field $h$ and inverse temperature $\beta$, $\mu_{h,\beta}$.

If $\mu_1$ and $\mu_2$ are F.K.G., $\mu_1$ precedes $\mu_2$ in the F.K.G. order if for any increasing event $E$

$$\mu_1(E) \leq \mu_2(E)\,. \tag{1.3}$$

It turns out that the measures $\mu_x$ and $\mu_{h,\beta}$ are respectively F.K.G.– increasing in $x$ and in $h$ (for fixed $\beta$).

In this paper we shall develope a scheme strongly analogous to the one summarized so far by considering, instead of the lattice structure of $\Omega_S$, its group structure. If we identify the elements of $\Omega_S$ with the subsets of $S$, the natural group operation can be defined as the symmetric difference between subsets (equivalently one could consider the multiplicative group structure of $\{-1,1\}^S$).

More precisely we propose the following definitions:

**Definition 1.1.** A probability measure on $\Omega_S$ is *G-regular* if for any two subgroups of $\Omega_S$, $G_1$ and $G_2$

$$\mu(G_1 \cap G_2) \geq \mu(G_1)\mu(G_2)\,. \tag{1.4}$$

**Definition 1.2.** The measure $\mu_1$ is *G-smaller than* $\mu_2$ if for any subgroup $G$ of $\Omega_S$

$$\mu_1(G) \leq \mu_2(G)\,. \tag{1.5}$$

**Definition 1.3.** The measure $\mu_1$ is *strongly G-smaller than* $\mu_2$ if for any pair of subgroups of $\Omega_S$, $G_1$ and $G_2$, such that $G_1 \subset G_2$

$$\frac{\mu_1(G_1)}{\mu_1(G_2)} \leq \frac{\mu_2(G_1)}{\mu_2(G_2)}\,. \tag{1.6}$$

Our main result is contained in the following theorem:

**Theorem 1.1.** *If $x \in [0, \frac{1}{2}]$ and $F$ is a subgroup of $\Omega_S$, the measures $\mu_x$ and $\mu_x(\ |F)$ are G-regular and strongly G-decreasing in $x$.*

Since it is easy to obtain some Gibbs measure (for example in the case of a zero-field n.n. Ising model) by conditioning to a subgroup the Bernoulli measure, Theorem 1.1 has some interesting consequences in Statistical Mechanics; in particular it implies the following corollary:

**Corollary 1.1.** *The measures $\mu_{0,\beta}$ are G-regular and strongly G-increasing in $\beta$.*

Corollary 1.1 can be considered as a generalization of the Griffiths' inequalities [3].

We remark that Baumgartner [4] recognized the relationship between the group structure of $\Omega_S$ and Griffiths' inequalities. Gruber, Hintermann and Merlini in their book [5] also exploited the natural group structure of $\Omega_S$. The direction of our work is, in a sense, complementary to the one in [5]: a large class of systems is there studied by using Statistical-Mechanical and group-theoretical tools, whereas our main result amounts in recognizing the simple property of the Bernoulli measure expressed by Theorem 1.1 as the basis of some Statistical-Mechanical techniques.

Sec. 2 contains our results concerning unconditioned Bernoulli measures; Bernoulli measures conditioned to a group are considered in Sec. 3, where the proof of Theorem 1.1 is completed by using a generalization of (1.2). Some applications to Statistical Mechanics (in particular to the Ising model and to gauge models) are in Sec. 4.

## 2. Regularity and monotonicity of Bernoulli measures

In this section we prove that the Bernoulli measues $\mu_x$ are G-regular and G-ordered for $x \in [0, \frac{1}{2}]$. We first prove that for any group $G$, $\mu_x(G)$ is a not increasing function of $x \in [0, \frac{1}{2}]$. Indeed we have a stronger result: $\mu_x(G)$ has derivatives of alternate sign in $[0, \frac{1}{2}]$. The G-regularity condition is proved at the end of the section.

We begin recalling the basic properties of the groups we use.

An element $\omega \in \Omega_S$ is a sequence of 0's and 1's on $S$ which we shall identify with the subset of $i \in S$ such that $\omega(i) = 1$. $\Omega_S$ is a group with respect to the operation of symmetric difference of two elements $\omega_1$ and $\omega_2$, that we denote $\omega_1 \cdot \omega_2$. One can compute the product of two configurations of 0's and 1's by using in each site the rule $0 \cdot 1 = 1 \cdot 0 = 1, 0 \cdot 0 = 1 \cdot 1 = 0$. The null configuration corresponding to the empty set is the identity of the group and so each element is its own inverse.

If $G$ is a subgroup of $\Omega_S$ the binary relation $\sim$ in $\Omega_S$ defined by $\omega_1 \sim \omega_2$ if and only if $\omega_1 \cdot \omega_2 \in G$ is an equivalence relation. The elements of the partition of $\Omega_S$ so generated are the cosets of the group $G$. The group itself is an element of the partition. Any coset $L$ different from $G$ is so disjoint from $G$ and is given by

$$L = \sigma \cdot G = \{\alpha \in \Omega_S \mid \alpha = \sigma \cdot \omega, \omega \in G\}$$

for any $\sigma \in L$. It is also easy to see that $G$ and $L$ have the same cardinality: $|G| = |L|$. If $H$ and $K$ are two cosets of the group $G$ we put

$$H \cdot K = \{\omega \in \Omega_S \mid \omega = \omega_1 \cdot \omega_2, \omega_1 \in H, \omega_2 \in K\}. \tag{2.1}$$

$H \cdot K$ is a coset of $G$. The set of the cosets of $G$ is a group with respect to the operation just defined. The identity is the group itself.

Let $\mu_{p_i}$ be the probability measure on $\Omega_i$, $i \in S$, defined by $\mu_{p_i}(1) = p_i$, $\mu_{p_i}(0) = 1 - p_i, p_i \in [0, 1]$. If $p = (p_i, i \in S)$, $\mu_p$ denotes the measure on $\Omega_S$ product of the $\mu_{p_i}$'s. In the following we shall continue to use the notation $\mu_x$, intoduced in sec. 1, for the measure $\mu_p$ where $p_i = x$, $\forall i \in S$.

If $\omega \in \Omega_S$, we denote by $\omega^c$ the complement of $\omega$ in $S$ and we put

$$p^\omega = \begin{cases} \prod_{i \in \omega} p_i, & \text{if } \omega \neq 0 \\ 1, & \text{if } \omega = 0 \end{cases}$$

where 0 denotes the null configuration. We have

$$\begin{aligned} \mu_p(\omega) &= p^\omega (1-p)^{\omega^c} \\ &= p^\omega (1 - 2p + p)^{\omega^c} \\ &= p^\omega \sum_{\sigma \subset \omega^c} (1-2p)^\sigma p^{\omega^c \setminus \sigma} \\ &= \sum_{\sigma \subset \omega^c} p^{S \setminus \sigma} (1-2p)^\sigma \end{aligned}$$

If $E \subset \Omega_S$

$$\mu_p(E) = \sum_{\omega \in E} \sum_{\sigma \subset \omega^c} p^{S\setminus\sigma}(1-2p)^\sigma$$

$$= \sum_{\sigma \subset S} p^{S\setminus\sigma}(1-2p)^\sigma |E_0^\sigma| \tag{2.2}$$

where we define if $\alpha \in \Omega_\sigma$, $\sigma \subset S$,

$$E_\alpha^\sigma = \{\omega \in \Omega_{S\setminus\sigma} \mid \omega\alpha \in E\}$$

where $\omega\alpha$ is the configuration of $\Omega_S$ that coincides with $\omega$ in $S\setminus\sigma$ and $\alpha$ in $\sigma$. In particular $E_0^\emptyset = E$ and

$$E_\alpha^S = \begin{cases} \emptyset & \text{if } \alpha \notin E \\ \Omega_\emptyset, & \text{if } \alpha \in E \end{cases}$$

where $|\Omega_\emptyset| = 1$, $\mu_p(\Omega_\emptyset) = 1$.

If $G$ is a subgroup of $\Omega_S$,$H$ and $K$ are two cosets of $G$ (which in particular can coincide with $G$) $\sigma \subset S$ and $\alpha, \beta \in \Omega_\sigma$ the following statements hold:

(2.3a) $G_0^\sigma$ is a subgroup of $\Omega_{S\setminus\sigma}$; all the $G_\alpha^\sigma$, $H_\alpha^\sigma$, $(H \cdot K)_\alpha^\sigma$ which are nonempty are cosets of $G_0^\sigma$ (in particular they have the same cardinality as $G_0^\sigma$);

(2.3b) $H_\alpha^\sigma \neq \emptyset, K_\beta^\sigma \neq \emptyset \Rightarrow H_\alpha^\sigma \cdot K_\beta^\sigma = (H \cdot K)_{\alpha\cdot\beta}^\sigma$;

(2.3c) $|G_0^\sigma||(H \cdot K)_{\alpha\cdot\beta}^\sigma| \geq |H_\alpha^\sigma||K_\beta^\sigma|$.

(2.3a) is a direct consequence of the definitions. In order to prove (2.3b) we note that $H_\alpha^\sigma \cdot K_\beta^\sigma \subset (H \cdot K)_{\alpha\cdot\beta}^\sigma$. By hypothesis both are nonempty; (2.3a) then implies that both are cosets of $G_0^\sigma$, so that the inclusion must hold as an equality.

In order to prove (2.3c) we note that if one of the sets $H_\alpha^\sigma$ and $K_\beta^\sigma$ is empty the inequality is trivially true. In the other case we apply (2.3a) and (2.3b) and (2.3c) follows as an equality.

If $G$ is a group and $H$ a coset of $G$ different from $G$, we remark that it is $\mu_0(G) = 1$, $\mu_0(H) = 0$ and $\mu_{\frac{1}{2}}(G) = \mu_{\frac{1}{2}}(H) = 2^{-|S|}|G|$.

**Lemma 2.1.** *If $G$ is a group, $H$ a coset of $G$ and $p \in [0, \frac{1}{2}]^S$ then*

$$\mu_p(G) \geq \mu_p(H)\,. \tag{2.4}$$

*Proof.* We use eq. (2.2) for $G$ and $H$ and we get

$$\mu_p(G) - \mu_p(H) = \sum_{\sigma \subset S} p^{S \setminus \sigma}(1-2p)^{\sigma}(|G_0^{\sigma}| - |H_0^{\sigma}|)\,.$$

By using (2.3a) we get $\forall \sigma, |G_0^{\sigma}| - |H_0^{\sigma}| \geq 0$ and this concludes the proof.

In the sequel we shall need the following simple consequence of Lemma 2.1:

**Lemma 2.2.** *If $G$ is a group, $H$ and $K$ cosets of $G$ and $p \in [0, \frac{1}{2}]^S$ then*

$$\mu_p(G) + \mu_p(H \cdot K) \geq \mu_p(H) + \mu_p(K)\,. \tag{2.4a}$$

*Proof.* This lemma can be deduced from the previous one simply remarking that if $H$ and $K$ are nonempty, $H \cup K$ is a coset of the group $G \cup (H \cdot K)$.

We remark that $\mu_p(E_\alpha^\sigma)$ does not depend on the $p_i$'s, $i \in \sigma$ and we define

$$\left(\frac{\partial}{\partial p}\right)^{\sigma} = \prod_{i \in \sigma} \frac{\partial}{\partial p_i}\,.$$

**Proposition 2.1.** *If $G$ is a subgroup of $\Omega_S$, $\sigma \subset S$ and $p \in [0, \frac{1}{2}]^S$ then*

$$\left(-\frac{\partial}{\partial p}\right)^{\sigma} \mu_p(G) \geq 0\,. \tag{2.5}$$

*In particular, if $p_i = x$, $x \in [0, \frac{1}{2}]$, for any $k$*

$$\left(-\frac{d}{dx}\right)^{k} \mu_x(G) \geq 0\,. \tag{2.6}$$

*Proof.* We first give a simple proof of (2.5) in the cases $\sigma = \{i\}$ and $\sigma = \{i, j\}$. This is enough to prove (2.6) for $k = 1, 2$; in the sequel the inequality (2.6) shall be used only for $k = 1$. We have

$$\mu_p(G) = \mu_p(G_0^i)(1-p_i) + \mu_p(G_1^i)p_i$$

$$\mu_p(G) = \mu_p(G_{00}^{ij})(1-p_j)(1-p_i) + \mu_p(G_{10}^{ij})p_i(1-p_j) + \mu_p(G_{01}^{ij})(1-p_i)p_j + \mu_p(G_{11}^{ij})p_ip_j$$

and

$$-\frac{\partial}{\partial p_i}\mu_p(G)=\mu_p(G_0^i)-\mu_p(G_1^i)$$

$$\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\mu_p(G)=\mu_p(G_{00}^{ij})+\mu_p(G_{11}^{ij})-\mu_p(G_{10}^{ij})-\mu_p(G_{01}^{ij}).$$

In the case $\sigma=\{i\}$ we use (2.3a) and (2.4). In the case $\sigma=\{i,j\}$ if $G_{10}^{ij}$ and $G_{01}^{ij}$ are nonempty we have from (2.3b)

$$G_{11}^{ij}=G_{10}^{ij}\cdot G_{01}^{ij}$$

and then we can use Lemma 2.2.

In the general case, from

$$\mu_p(G)=\sum_{\alpha\subset\sigma}p^{\alpha}(1-p)^{\sigma\setminus\alpha}\mu_p(G_\alpha^\sigma)$$

it easily follows

$$\left(-\frac{\partial}{\partial p}\right)^{\sigma}\mu_p(G)=\sum_{\alpha\subset\sigma}(-1)^{|\alpha|}\mu_p(G_\alpha^\sigma).$$

We notice that some of the $G_\alpha^\sigma$'s can be empty, but if they are not, they are cosets of $G_0^\sigma$. We apply (2.2) to $G_\alpha^\sigma$:

$$\mu_p(G_\alpha^\sigma)=\sum_{\gamma\subset S\setminus\sigma}p^{(S\setminus\sigma)\setminus\gamma}(1-2p)^{\gamma}|(G_\alpha^\sigma)_0^\gamma|.$$

We get, interchanging the order of summation on $\alpha$ and $\gamma$

$$\left(-\frac{\partial}{\partial p}\right)^{\sigma}\mu_p(G)=\sum_{\gamma\subset S\setminus\sigma}p^{(S\setminus\sigma)\setminus\gamma}(1-2p)^{\gamma}\sum_{\alpha\subset\sigma}(-1)^{|\alpha|}|(G_0^\gamma)_\alpha^\sigma|$$

where we have used that $(G_\alpha^\sigma)_0^\gamma=(G_0^\gamma)_\alpha^\sigma$ as $\sigma\cap\gamma=\emptyset$. In order to prove (2.5) it suffices to prove that $\forall\sigma\subset S$

$$\sum_{\alpha\subset\sigma}(-1)^{|\alpha|}|F_\alpha^\sigma|\geq 0 \tag{2.7}$$

where we have put $F=G_0^\gamma$. The set $A=\{\alpha\subset\sigma\,|\,F_\alpha^\sigma\neq\emptyset\}$ is a subgroup of $\Omega_\sigma$ since $F_0^\sigma$ is a group and

$$F_{\alpha_1}^\sigma\neq\emptyset,\ F_{\alpha_2}^\sigma\neq\emptyset\Rightarrow F_{\alpha_1\cdot\alpha_2}^\sigma\neq\emptyset.$$

If $A=\{0\}$ (2.7) is trivially true. If $|A|\geq 2$ we put $A_+=\{\alpha\in A\,|\,|\alpha|$ is even$\}$ and $A_-=A\setminus A_+$. If $A_-=\emptyset$ (2.7) is again trivially true. If $A_-\neq\emptyset$, then $A_-$ is a coset of $A_+$; hence $|A_+|=|A_-|$; since all the sets $F_\alpha^\sigma$, $\alpha\in A$ have the same cardinality, we get

$$\sum_{\alpha\in A_+}|F_\alpha^\sigma|=\sum_{\alpha\in A_-}|F_\alpha^\sigma|$$

and this concludes the proof of eq. 2.7.

In order to prove (2.6) we use

$$-\frac{d}{dx} = \sum_{i \in S} -\frac{\partial}{\partial p_i}\bigg|_{p_i = x}$$

$$\left(-\frac{d}{dx}\right)^k = \sum_{(i_1, \ldots, i_k) \in S^k} -\frac{\partial}{\partial p_{i_1}} \cdots -\frac{\partial}{\partial p_{i_k}}\bigg|_{p_i = x}.$$

The sequences $(i_1, \ldots, i_k)$ with at least one overlapping give a null contribution to the sum because for any event $E$

$$\frac{\partial^n}{\partial p_i^n} \mu_p(E) = 0, \forall n \geq 2.$$

Hence

$$\left(-\frac{d}{dx}\right)^k \mu_x(G) = k! \sum_{\sigma \subset S, |\sigma| = k} \left(-\frac{\partial}{\partial p}\right)^\sigma \mu_p(G)\bigg|_{p_i = x}$$

and (2.5) implies (2.6).

The following proposition is a weaker version of an inequality which we shall prove in the next section. Nevertheless we give here an independent proof of it because we think that the simple proof given here is more transparent and it could serve as an illustration of the intuitive meaning of the order we have introduced between measures.

**Proposition 2.2.** *If $F$ and $G$ are two subgroups of $\Omega_S$ and $x \in [0, \frac{1}{2}]$ then*

$$\mu_x(F \cap G) \geq \mu_x(F)\mu_x(G).$$

In order to prove this proposition we need the following definitions. If $\{i, j\} \in S$ let $\tilde{\mu}_x$ be the restriction of $\mu_x$ to $\Omega_{S \setminus \{i,j\}}$ and let $\nu_x$ be the probability measure on $\Omega_{ij}$ defined by

$$\nu_x(0,0) = 1 - x,\ \nu_x(1,1) = x,\ \nu_x(0,1) = \nu_x(1,0) = 0.$$

We define $\mu_x^{ij}$ the product measure $\tilde{\mu}_x \times \nu_x$. If $A$ and $A'$ are disjoint subsets mapped by a one to one mapping, one can naturally define, using the above definition, the measure $\mu_x^{A,A'}$.

**Lemma 2.3.** *If $G$ is a subgroup of $\Omega_S$ and $x \in [0, \frac{1}{2}]$ then*

(2.8) $\quad \mu_x^{ij}(G) \geq \mu_x(G).$

*Proof.* We have

$$\mu_x(G) = (1-x)^2 \mu_x(G_{00}^{ij}) + x(1-x)[\mu_x(G_{10}^{ij}) + \mu_x(G_{01}^{ij})] + x^2 \mu_x(G_{11}^{ij})$$

$$\mu_x^{ij}(G) = (1-x)\mu_x(G_{00}^{ij}) + x\mu_x(G_{11}^{ij})$$

$$\mu_x^{ij}(G) - \mu_x(G) = x(1-x)[\mu_x(G_{00}^{ij}) + \mu_x(G_{11}^{ij}) - \mu_x(G_{01}^{ij}) - \mu_x(G_{10}^{ij})]$$

and, using Lemma 2.2 the lemma follows.

It is easy to convince oneself that one can extend this lemma to the measure $\mu_x^{A,A'}$: if $G$ is a group

$$\mu_x^{A,A'}(G) \geq \mu_x(G)\,. \tag{2.9}$$

In order to prove Proposition 2.2, we consider a copy $S'$ of $S$, the Bernoulli measure $\mu'_x$ on $\Omega_{S'}$, copy of $\mu_x$, and the measure $\bar{\mu}_x$ on $\Omega_S \times \Omega_{S'}$ given by $\mu_x \times \mu'_x$. Given the two subgroups $F$ and $G$, if $F'$ is the copy of $F$ in $\Omega_{S'}$, $G \times F'$ is a subgroup of $\Omega_S \times \Omega_{S'}$ and one obviously gets

$$\mu_x(G)\mu_x(F) = \bar{\mu}_x(G \times F')\,.$$

It is easy to check that

$$\mu_x(G \cap F) = \bar{\mu}_x^{S,S'}(G \times F')\,.$$

Inequality (2.9) for the group $G \times F'$ completes the proof.

## 3. The case of conditioned Bernoulli measures

In this section we complete the proof of Theorem 1.1 by proving that the Bernoulli measures conditioned to a group are G-regular and G-ordered. These properties are both a consequence of the following proposition.

**Proposition 3.1.** *Let $p \in [0, \frac{1}{2}]^S$. If $G$ is a subgroup of $\Omega_S$ and $H$ and $K$ are two cosets of $G$, then*

$$\mu_p(G)\mu_p(H \cdot K) \geq \mu_p(H)\mu_p(K)\,; \tag{3.1}$$

*if $G_1$ and $G_2$ are subgroups, then*

$$\mu_p(G_1 \cdot G_2)\mu_p(G_1 \cap G_2) \geq \mu_p(G_1)\mu_p(G_2)\,. \tag{3.2}$$

If $G$ is a group and $p \in [0, \frac{1}{2}]^S$ we put $\forall \sigma \subset S$

$$v_p^G(\sigma) = \frac{p^{S \setminus \sigma}(1-2p)^\sigma |G_0^\sigma|}{\mu_p(G)}\,. \tag{3.3}$$

Eq. (2.2) implies that $v_p^G$ is a probability measure on $\Omega_S$.

**Lemma 3.1.** *The measure $v_p^G$ is F.K.G.*

*Proof.* As we remarked, it is enough to prove that the measure $\nu_p^G$ satisfies the inequality (1.2), and for this we only need to prove that for any group $G$

(3.4) $\quad |G_0^{\sigma_1 \cup \sigma_2}||G_0^{\sigma_1 \cap \sigma_2}| \geq |G_0^{\sigma_1}||G_0^{\sigma_2}|.$

We first consider the case $\sigma_1 \cap \sigma_2 = \emptyset$. Then $G_0^{\sigma_1 \cap \sigma_2} = G$ and so we need to prove

(3.5) $\quad |G_0^{\sigma_1 \cup \sigma_2}||G| \geq |G_0^{\sigma_1}||G_0^{\sigma_2}|.$

We have

$$|G_0^{\sigma_1}| = \sum_{\alpha_2 \subset \sigma_2} |G_{0\,\alpha_2}^{\sigma_1 \sigma_2}|$$

$$|G_0^{\sigma_2}| = \sum_{\alpha_1 \subset \sigma_1} |G_{\alpha_1 0}^{\sigma_1 \sigma_2}|$$

$$|G| = \sum_{\alpha_1 \subset \sigma_1} \sum_{\alpha_2 \subset \sigma_2} |G_{\alpha_1 \alpha_2}^{\sigma_1 \sigma_2}|$$

where some of the $G$'s can be empty. The ones that are not empty are cosets of the group $G_{0\,0}^{\sigma_1 \sigma_2}$ and applying (2.3b) and (2.3c) we get $\forall \alpha_1, \alpha_2$

$$G_{\alpha_1 0}^{\sigma_1 \sigma_2} \neq \emptyset,\ G_{0\,\alpha_2}^{\sigma_1 \sigma_2} \neq \emptyset \ \Rightarrow\ G_{\alpha_1 \alpha_2}^{\sigma_1 \sigma_2} = G_{\alpha_1 0}^{\sigma_1 \sigma_2} \cdot G_{0\,\alpha_2}^{\sigma_1 \sigma_2} \neq \emptyset$$

and

(3.6) $\quad |G_{\alpha_1 0}^{\sigma_1 \sigma_2}||G_{0\,\alpha_2}^{\sigma_1 \sigma_2}| \leq |G_{0\,0}^{\sigma_1 \sigma_2}||G_{\alpha_1 \alpha_2}^{\sigma_1 \sigma_2}|.$

Using this inequality we get

$$\begin{aligned} |G_0^{\sigma_1}||G_0^{\sigma_2}| &= \sum_{\alpha_1 \subset \sigma_1} \sum_{\alpha_2 \subset \sigma_2} |G_{0\,\alpha_2}^{\sigma_1 \sigma_2}||G_{\alpha_1 0}^{\sigma_1 \sigma_2}| \\ &\leq \sum_{\alpha_1 \subset \sigma_1} \sum_{\alpha_2 \subset \sigma_2} |G_{0\,0}^{\sigma_1 \sigma_2}||G_{\alpha_1 \alpha_2}^{\sigma_1 \sigma_2}| \\ &= |G_{0\,0}^{\sigma_1 \sigma_2}||G|. \end{aligned}$$

Since $G_0^{\sigma_1 \cup \sigma_2} = G_{0\,0}^{\sigma_1 \sigma_2}$, we get (3.5).

We now consider the case $\sigma_1 \cap \sigma_2 = \tau \neq \emptyset$. We apply (3.5) to $G_0^\tau$ in place of $G$, and to $\tau_1 = \sigma_1 \backslash \tau$, $\tau_2 = \sigma_2 \backslash \tau$, as $\tau_1 \cap \tau_2 = \emptyset$. We get

$$|G_{0\,0}^{\tau \tau_1 \cup \tau_2}||G_0^\tau| \geq |G_{0\,0}^{\tau \tau_1}||G_{0\,0}^{\tau \tau_2}|.$$

We notice that $G_0^\tau = G_0^{\sigma_1 \cap \sigma_2}$, $G_{0\,0}^{\tau \tau_1 \cup \tau_2} = G_0^{\sigma_1 \cup \sigma_2}$, $G_{0\,0}^{\tau \tau_1} = G_0^{\sigma_1}$, $G_{0\,0}^{\tau \tau_2} = G_0^{\sigma_2}$ and this completes the proof of the lemma.

*Proof of Proposition 3.1.* We first prove eq. (3.1). From eq. (2.2) and (3.3) it is

$$\frac{\mu_p(H \cdot K)}{\mu_p(G)} = \sum_{\sigma \subset S} \nu_p^G(\sigma) \frac{|(H \cdot K)_0^\sigma|}{|G_0^\sigma|}.$$

Using the inequality

(3.7) $\quad |(H \cdot K)_0^\sigma||G_0^\sigma| \geq |H_0^\sigma||K_0^\sigma|$

which is again a particular case of (2.3c), we get

$$\frac{\mu_p(H\cdot K)}{\mu_p(G)} \geq \sum_{\sigma\subset S} \nu_p^G(\sigma)\frac{|H_0^\sigma||K_0^\sigma|}{|G_0^\sigma||G_0^\sigma|}$$
$$= \sum_{\sigma\subset S} \nu_p^G(\sigma)\chi_H(\sigma)\chi_K(\sigma)$$
$$= E_\nu(\chi_H\chi_K),$$

where

$$\text{(3.8)} \qquad \chi_H(\sigma) = \frac{|H_0^\sigma|}{|G_0^\sigma|}$$

and $E_\nu$ denotes the expectation with respect to $\nu_p^G$. The functions $\chi_H$ and $\chi_K$, that take only the values 0 and 1, are both decreasing in the order by inclusion of the subsets of $S$. As recalled in Sec. 1, this is sufficient by [2] to conclude that

$$E_\nu(\chi_H\chi_K) \geq E_\nu(\chi_H)E_\nu(\chi_K).$$

The proof of (3.1) is completed by observing that

$$E_\nu(\chi_H) = \frac{\mu_p(H)}{\mu_p(G)}.$$

We now prove inequality (3.2). Put $G = G_1 \cap G_2$. By Def. (3.3)

$$\frac{\mu_p(G_1\cdot G_2)}{\mu_p(G)} = \sum_{\sigma\subset S} \nu_p^G(\sigma)\frac{|(G_1\cdot G_2)_0^\sigma|}{|G_0^\sigma|}$$
$$\geq \sum_{\sigma\subset S} \nu_p^G(\sigma)\frac{|(G_1)_0^\sigma\cdot(G_2)_0^\sigma|}{|G_0^\sigma|},$$

where we have used the inclusion, which easily follows from the definitions,

$$\text{(3.9)} \qquad (G_1\cdot G_2)_0^\sigma \supset (G_1)_0^\sigma\cdot(G_2)_0^\sigma.$$

We now remark that for any two groups $F$ and $G$

$$\text{(3.10)} \qquad |F\cdot G||F\cap G| = |F||G|.$$

If $F\cap G = \{0\}$, $|F\cdot G| = |F||G|$ and the equation is trivially true. In general the remark can be proved by applying the previous equality to the quotient groups of $F$, $G$ and $F\cdot G$ with respect to $F\cap G$.

Using (3.10) and the easy equality $(G_1)_0^\sigma\cap(G_2)_0^\sigma = G_0^\sigma$ we get

$$\frac{\mu_p(G_1\cdot G_2)}{\mu_2(G)} \geq \sum_{\sigma\subset S} \nu_p^G(\sigma)\frac{|(G_1)_0^\sigma|}{|G_0^\sigma|}\frac{|(G_2)_0^\sigma|}{|G_0^\sigma|}.$$

We put

$$\eta_1(\sigma) = \frac{|(G_1)_0^\sigma|}{|G_0^\sigma|}$$

and we observe that $\eta_1$ is a decreasing function of $\sigma$. It suffices to prove that if $\sigma' = \sigma \cup \{i\}$, $i \in S$, then $\eta_1(\sigma') \leq \eta_1(\sigma)$. For this it is enough to prove the inequality

$$(3.11)\qquad |G_0^\sigma||(G_1)_0^{\sigma'}| \leq |G_0^{\sigma'}||(G_1)_0^\sigma| .$$

By using the inequalities

$$|(G_1)_0^\sigma| = |(G_1)_0^{\sigma'}| + |(G_1)_{01}^{\sigma i}|$$

$$|G_0^\sigma| \quad = |G_0^{\sigma'}| + |G_{01}^{\sigma i}|$$

the inequality (3.11) can be written

$$(3.12)\qquad |(G_1)_0^{\sigma'}||G_{01}^{\sigma i}| \leq |(G_1)_{01}^{\sigma i}||G_0^{\sigma'}| .$$

If $G_{01}^{\sigma i} = \emptyset$ the equation is trivially true. Suppose $G_{01}^{\sigma i} \neq \emptyset$. Then $G_{01}^{\sigma i}$ is a coset of $G_{00}^{\sigma i} = G_0^{\sigma'}$, and so it has the same cardinality. Furthermore, since $G \subset G_1$, we have also $(G_1)_{01}^{\sigma i} \neq \emptyset$; hence $(G_1)_{01}^{\sigma i}$ is a coset of $(G_1)_{00}^{\sigma i} = (G_1)_0^{\sigma'}$ so that (3.12) holds as an equality. This proves the observation.

The proof of (3.2) can now be achieved exactly as the one of (3.1).

We observe that (3.2) is at same time an improvement of Proposition 2.2 and a generalization of the inequality (1.2) for Bernoulli measures. In fact the cylinder obtained by putting equal to zero all the spins in a given subset $\sigma$ of $S$ is a subgroup of $\Omega_S$; if the subgroups $G_1$, $G_2$ are obtained in this way from the subsets $\sigma_1$, $\sigma_2$ the inequality (3.2) reduces to (1.2).

**Proposition 3.2.** *If F and G are groups and H is a coset of G,* $\forall p \in [0, \frac{1}{2}]^S$

$$(3.13)\qquad \frac{\partial}{\partial p_i} \frac{\mu_p(H)}{\mu_p(G)} \geq 0 ,$$

$$(3.14)\qquad \frac{\partial}{\partial p_i} \mu_p(G|F) \leq 0 .$$

*Proof.* From

$$\mu_p(G) = p_i \mu_p(G_1^i) + (1 - p_i)\mu_p(G_0^i)$$

and analogous equation for $\mu_p(H)$, we get, performing the derivative,

$$\frac{\partial}{\partial p_i} \frac{\mu_p(H)}{\mu_p(G)} = \mu_p(G)^{-2}[\mu_p(G_0^i)\mu_p(H_1^i) - \mu_p(H_0^i)\mu_p(G_1^i)] .$$

We can apply (3.1) because $H_1^i = H_0^i \cdot G_1^i$ and this proves eq. 3.13.

Inequality (3.14) follows easily from inequality (3.13) and the remark that there are $n$ cosets of $G \cap F$, $H_1, \ldots, H_n$, such that

$$F = (G \cap F) \cup \bigcup_{i=1}^{n} H_i ,$$

and so

$$\mu_p(F) = \mu_p(G \cap F) + \sum_{i=1}^{n} \mu_p(H_i) .$$

*Proof of Theorem 1.1.* The G-regularity of the measure $\mu_x$ was already proved in sec. 2. If $F$ is a subgroup of $\Omega_S$, by using inequality (3.2), we get

$$\mu_x(G_1 \cap G_2 | F) \geq \mu_x(G_1 | F) \mu_x(G_2 | F) \frac{\mu_x(F)}{\mu_x((G_1 \cap F) \cdot (G_2 \cap F))} .$$

Since $(G_1 \cap F) \cdot (G_2 \cap F)$ is a subgroup of $F$, we get the G-regularity of $\mu_x(\ |F)$. The G-monotonicity of $\mu_x$ is a particular case of Proposition 2.1. The strong G-monotonicity of $\mu_x$ and $\mu_x(\ |F)$ is a direct consequence of inequality (3.14).

## 4. Applications to statistical mechanics

The aim of this section is to show that some Statistical Mechanical inequalities are a particular case of the results of the preceding section.

We first consider the nearest neighbour (n.n.) Ising model at zero magnetic field and $+1$ boundary conditions; for sake of simplicity we recall the definitions we need in the two dimensional case. The space of configurations is $\{-1, 1\}^{\Lambda}$ where $\Lambda$ is a square subset of $Z^2$ and the spins on the sites of $\delta\Lambda$, the external boundary of $\Lambda$, are put to be equal to 1. For any $s \in \{-1, 1\}^{\Lambda}$ a contour configuration $\gamma_s$ can be defined in the following way. For any pair of n.n. we consider the unit bond that separates them; if we denote it by $c$, the pair is denoted by $\{c_1, c_2\}$. We put

$$C = \{c | \{c_1, c_2\} \cap \Lambda \neq \emptyset\} ;$$

$\gamma_s$ is the element of $\{0, 1\}^C$ given by

$$\text{(4.1)} \qquad \gamma_s(c) = \begin{cases} 0, & \text{if } s(c_1) = s(c_2) \\ 1, & \text{if } s(c_1) \neq s(c_2) . \end{cases}$$

The set of contour configurations is

$$\text{(4.2)} \qquad \Gamma = \{\gamma \in \{0, 1\}^C | \gamma = \gamma_s \text{ for some } s \in \{0, 1\}^{\Lambda}\} .$$

It is easy to recognize that, denoting by $st$ the pointwise ordinary product of the spin configurations $s$ and $t$ (while the dot denotes the symmetric difference) one has

$$\text{(4.3)} \qquad \gamma_s \cdot \gamma_t = \gamma_{st} .$$

Equality (4.3) immediately implies that $\Gamma$ is a subgroup of $\{0, 1\}^C$. The Gibbs measure $\mu^+_{0,\beta}$ is defined assigning to any $s \in \{-1, 1\}^{\Lambda}$ the weight

$$e^{\beta \sum_{c \in C} s(c_1) s(c_2)} .$$

This weight is proportional to

$$e^{-2\beta |\gamma_s|}$$

where $|\gamma_s|$ is the cardinality of $\gamma_s$. Hence $\mu^+_{0,\beta}$ is a Bernoulli measure of parameter $x = (1+e^{-2\beta})^{-1}e^{-2\beta} \in [0, 1/2]$ conditioned to the subgroup $\Gamma$ and the results of the preceding section can be applied.

As an example, let us consider for any $A \subset \Lambda$ the set of contour configurations $\Gamma_A$, corresponding to the set of spin configurations $E_A = \{s \mid s(i) = s(j) \forall i, j \in A\}$. By using (4.3) it is immediate to see that $\Gamma_A$ is a subgroup of $\Gamma$. Since

$$\mu^+_{0,\beta}(E_A) = \mu_x(\Gamma_A \mid \Gamma)\,, \tag{4.4}$$

the G-monotonicity of $\mu_x(\ \mid \Gamma)$ reduces in this case to the well known monotonicity in $\beta$ of $\mu^+_{0,\beta}(E_A)$. By using the strong G-monotonicity of $\mu_x(\ \mid \Gamma)$ we get the monotonocity in $\beta$ of the ratios $\mu^+_{0,\beta}(E_A)/\mu^+_{0,\beta}(E_B)$ for any $B \supset A$. The same arguments work if one considers periodic or open boundary conditions.

As another example we consider the Gibbs measure $\mu^J_{0,\beta}$ defined by the weight

$$e^{\beta \sum_{b\in B} J_b s(b_1)s(b_2)}\,,$$

where $\Lambda$ be a finite set, $B = \{b \subset \Lambda \mid b = \{b_1, b_2\}\}$ and $J = (J_b, b \in B)$ is any vector with $J_b \geq 0$. We now show how the Griffiths' inequalities are a particular case of Proposition (3.1). We consider a subset $A$ of $\Lambda$ and let $s_A = \prod_{i\in A} s(i)$; there is a nice representation of $\langle s_A \rangle_J$, the expectation of $s_A$ with respect to the Gibbs measure, in terms of the Bernoulli measure $\mu_p$ where $p_b = e^{-\beta J_b}\sinh \beta J_b \in [0, 1/2]$, [5, 7]: if one defines $\forall n \in \{0,1\}^B$

$$\begin{aligned}
\partial n &= \{i \in \Lambda \mid \sum_{i\in b} n(b) \text{ is odd}\}\\
G &= \{n \in \{0,1\}^B \mid \partial n = \emptyset\}\\
H_A &= \{n \in \{0,1\}^B \mid \partial n = A\}
\end{aligned}$$

then

$$\langle s_A \rangle_J = \frac{\mu_p(H_A)}{\mu_p(G)}\,. \tag{4.5}$$

It is easy to see that $G$ is a group and that $H_A$ is one of its cosets. Furthermore if $A$ and $B$ are disjoint we have

$$\langle s_A s_B \rangle_J = \frac{\mu_p(H_{A\cup B})}{\mu_p(G)} \tag{4.6}$$

and it is easy to see that $H_{A\cup B} = H_A \cdot H_B$. We apply (3.1) and get the Griffiths' inequalities

$$\langle s_A s_B \rangle_J \geq \langle s_A \rangle_J \langle s_A \rangle_J\,.$$

In the gauge theories on the lattice the following model is considered [6]. For each bond $b$ of a $d$-dimensional, say $d = 3$, square lattice it is associated a spin variable

$\sigma_b \in \{-1, 1\}$, considered as an element of the group $Z_2$. The measure is defined by the weight

$$\exp \beta \sum_P \prod_{b \in P} \sigma_b$$

where $P$ runs over all the plaquettes. This measure can be obtained from a Bernoulli measure on the plaquettes variables $\omega_P \in \{-1, 1\}$, conditioned to the group $\bigcap_Q \{\prod_{P \in Q} \omega_P = 1\}$, where $Q$ runs over the cubes of the lattice. Hence also to this model the results of the preceding section apply exactly as in the previous cases.

**Acknowledgments.** We thank C. Maes for pointing out to us the paper by Baumgartner and for some interesting discussions.

## References

[1] Harris, T.E.: A lower bound for the critical probability in a certain percolation process.Proc. Camb. Phil. Soc. **59**, 13–20 (1960)
[2] Fortuin, C., Kasteleyn, P.W., Ginibre, J.: Correlation Inequalities on Some Partially Ordered Sets. Comm. Math. Phys. **22**, 89–103 (1971)
[3] Griffiths, R.: Correlation in ferromagnets. I and II. J. Math. Phys. **8**, 478–489 (1967)
[4] Baumgartner, B.: On the group structure, GKS and FKG inequalities for Ising model. J. Math. Phys. **34**, 2197–2199 (1983)
[5] Gruber, C., Hintermann, A., Merlini, D.: Group Analysis of Classical Lattice Systems. Lecture Notes in Physics 60. Springer-Verlag 1977
[6] Osterwalder, K.: Gauge theories on the lattice. Proc. of the 1976 Cargese summer school. Plenum Press 1977
[7] Aizenman, M.: Geometric Analysis of $\Phi^4$ Fields in Ising Models. Parts I and II. Comm. Math. Phys. **86**, 1–48 (1982)

Received July 19, 1990

Camillo Cammarota, Universita' di Roma La Sapienza, Dipartimento di Matematica, p.le A. Moro, 00185 Roma, Italia
Lucio Russo, Universita' di Roma Tor Vergata, Dipartimento di Matematica, via O. Raimondo, 00173 Roma, Italia